Matrices

Addition of Matrices

Matrices are rectangular arrays of elements in columns and rows. Algebraic addition of matrices is possible if they have the same shape, in which case (as for numbers) **A** + **B** = **B** + **A** and **A** - **B** = - (**B** - **A**). The algebraic sum of two matrices of the same shape is a matrix in which each element is the algebraic sum of corresponding elements in the two matrices. (Note: each element is identified by subscript numbers in the form x_{pq}, where p = row number and q = column number)

For 3 × 3 matrices, if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ then $\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$

Examples: $\begin{pmatrix} 2 & 0 & 5 \\ 3 & -1 & 4 \\ -3 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 7 & 3 & -1 \\ 2 & -2 & 1 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 4 \\ 5 & -3 & 5 \\ -2 & 5 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 5 & 2 \\ -4 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 2 \\ -3 & -4 \end{pmatrix}$

The application of a scalar multiple to a matrix means that each element of the matrix is multiplied by the same factor:

Example: If $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$ then $2\mathbf{A} + 3\mathbf{B} = \begin{pmatrix} -4 & 2 \\ 6 & -2 \end{pmatrix} + \begin{pmatrix} -9 & 6 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} -13 & 8 \\ 9 & 1 \end{pmatrix}$

Multiplication of matrices

This is only possible if the number of columns in the first matrix is the same as the number of rows in the second matrix. Each element of a matrix product is the sum of two products as shown below.

For 2 × 2 matrices, if $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} a_{11}\times b_{11} + a_{12}\times b_{21} & a_{11}\times b_{12} + a_{12}\times b_{22} \\ a_{21}\times b_{11} + a_{22}\times b_{22} & a_{21}\times b_{12} + a_{22}\times b_{22} \end{pmatrix}$

Example: If $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -10 & 5 \end{pmatrix}$

This is not the same as multiplication of numbers, as **AB** ≠ **BA**

Example: If $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$ then $\mathbf{BA} = \begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 12 & -5 \\ 1 & 0 \end{pmatrix}$

A matrix **A** can have an inverse $\mathbf{A}^{-1}$ only if it is a square matrix (ie number of rows and columns is the same)

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Example: If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ then $\mathbf{A}^{-1} = \dfrac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$

Note: $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and **AI = IA = A** , where **I** is the 2 × 2 identity matrix: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note also that if, for any given 2 x 2 matrix, $ad - bc = 0$, then an inverse does not exist

Using matrices to solve simultaneous equations

This method of solution is useful when there are three or more unknowns, but here is an example with two unknowns:

Given the equations: $\begin{matrix} 5x - 3y = 1 \\ 3x + 2y = 12 \end{matrix}$ write them in matrix form: $\begin{pmatrix} 5 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$ and then multiply both sides of this equation by the **inverse** of the matrix: $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{19}\begin{pmatrix} 2 & 3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \end{pmatrix} = \dfrac{1}{19}\begin{pmatrix} 38 \\ 57 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \therefore x = 2,\ y = 3$

Exercises

For the matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$ calculate the following:

1) **A + B** 2) **A - B** 3) **A +** 2**B** 4) 2**A + B** 5) **AB**

6) **BA** 7) $\mathbf{A}^{-1}$ 8) $\mathbf{B}^{-1}$ 9) $(\mathbf{AB})^{-1}$ 10) $(\mathbf{BA})^{-1}$

Use matrices to solve the simultaneous equations:

11) $3x + 2y = -5, \quad x - 4y = -4$ 12) $2x + 3y = 5, \quad 5x + 8y = 12$

Answers (given as a list of elements of a 2 × 2 matrix in order $x_{11}, x_{12}, x_{21}, x_{22}$)

1. (5, -1, 5, -3) 2. (-3, 5, 1, -5) 3. (9, -4, 7, -2) 4. (6, 1, 8, -7) 5. (8, -1, 4, -13) 6. (-5, 20, 5, 0)

7. (0.4, 0.2, 0.3, -0.1) 8. (0.1, 0.3, -0.2, 0.4) 9. (0.13, -0.01, 0.04, -0.08) 10. (0.05, -0.2, -0.05, 0)

11. $x = -2,\ y = 0.5$ 12. $x = 4,\ y = -1$