Calculus Solutions

1 $f(x)=\sqrt{\frac{x^3}{2}-\frac{2}{x^3}}=\left(\tfrac{1}{2}x^3-2x^{-3}\right)^{\frac{1}{2}}=g\left(h(x)\right)$, where $h(x)=\tfrac{1}{2}x^3-2x^{-3}\ \therefore h'(x)=\tfrac{3}{2}x^2+6x^{-4}$

and $g'\left(h(x)\right)=\tfrac{1}{2}\left(\tfrac{1}{2}x^3-2x^{-3}\right)^{-\frac{1}{2}}\ \therefore\ f'(x)=g'\left(h(x)\right)h'(x)=\tfrac{1}{2}\left(\tfrac{1}{2}x^3-2x^{-3}\right)^{-\frac{1}{2}}\left(\tfrac{3}{2}x^2+6x^{-4}\right)=\dfrac{3\left(x^2+\frac{4}{x^4}\right)}{4\sqrt{\frac{x^3}{2}-\frac{2}{x^3}}}$

2 $f(x)=\dfrac{e^{7x}}{x^2}\quad\therefore f'(x)=\left(7x^2e^{7x}-2xe^{7x}\right)=\dfrac{e^{7x}(7x-2)}{x^3}$

3 $f(x)=3x^2\ln x\quad\therefore f'(x)=3x^2\times\dfrac{1}{x}+6x\ln x=3x(2\ln x+1)=3x(\ln x^2+1)$

4 $f(x)=\ln(6x)\cos(2x)\quad\therefore f'(x)=-2\ln(6x)\sin(2x)+\dfrac{1}{x}\cos(2x)$

$\therefore f''(x)=-4\ln(6x)\cos(2x)-\dfrac{2}{x}\sin(2x)-\dfrac{2}{x}\sin(2x)-\dfrac{1}{x^2}\cos(2x)=-\cos(2x)\left(4\ln(6x)+\dfrac{1}{x^2}\right)-\dfrac{4}{x}\sin(2x)$

5 $f(x)=\dfrac{\ln(x-1)}{\cos x}\quad\therefore f'(x)=\dfrac{1}{\cos^2 x}\left(\dfrac{\cos x}{x-1}+\sin x\ln(x-1)\right)=\sec x\left(\dfrac{1}{x-1}+\tan x\ln(x-1)\right)$

6 $f(x)=\tan^{-1}\left(\dfrac{1}{x}\right)=g\left(h(x)\right)$, where $h(x)=\dfrac{1}{x}\ \therefore h'(x)=-\dfrac{1}{x^2}$ and $g'\left((h(x)\right)=\dfrac{1}{1+\left(\frac{1}{x}\right)^2}=\dfrac{x^2}{1+x^2}$

$\therefore f'(x)=g'\left(h(x)\right)h'(x)=\dfrac{x^2}{1+x^2}\times-\dfrac{1}{x^2}=-\dfrac{1}{1+x^2}$

7

$f(x)=\sin^{-1}\left(\sqrt{x}\right)=g\left(h(x)\right)$ where $h(x)=\sqrt{x}=x^{\frac{1}{2}}\ \therefore h'(x)=\tfrac{1}{2}x^{-\frac{1}{2}}$ and $g'\left(h(x)\right)=\dfrac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}=\dfrac{1}{\sqrt{1-x}}$

$\therefore f'(x)=g'\left(h(x)\right)h'(x)=\dfrac{1}{\sqrt{1-x}}\times\tfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{1}{\sqrt{1-x}}\times\dfrac{1}{2\sqrt{x}}=\dfrac{1}{2\sqrt{x(1-x)}}$

8

$f(x)=\sin^{-1}\left(\dfrac{1}{\sqrt{x}}\right)=g\left(h(x)\right)$ where $h(x)=\dfrac{1}{\sqrt{x}}=x^{-\frac{1}{2}}\ \therefore h'(x)=-\tfrac{1}{2}x^{-\frac{3}{2}}$ and $g'\left(h(x)\right)=\dfrac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x}}\right)^2}}=\dfrac{1}{\sqrt{1-\frac{1}{x}}}$

$\therefore f'(x)=g'\left(h(x)\right)h'(x)=\dfrac{1}{\sqrt{1-\frac{1}{x}}}\times-\tfrac{1}{2}x^{-\frac{3}{2}}=\dfrac{1}{\sqrt{1-\frac{1}{x}}}\times-\dfrac{1}{2\sqrt{x^3}}=-\dfrac{1}{2\sqrt{x^3(1-\frac{1}{x})}}=-\dfrac{1}{2\sqrt{x^2(x-1)}}-\dfrac{1}{2x\sqrt{x-1}}$

9 $f(x)=x^5e^{1+x^6}$; let $u=1+x^6\ \therefore\ldots du\equiv\ldots 6x^5dx$; $\quad\int x^5e^{1+x^6}dx\equiv\tfrac{1}{6}\int e^u du=\tfrac{1}{6}e^{1+x^6}+c$

10 $f(x)=x^3\cos\left(1+x^4\right)$; let $u=1+x^4\ \therefore \dots du\equiv\dots 4x^3dx$; $\int x^3\cos\left(1+x^4\right)dx\equiv\frac{1}{4}\int\cos u\,du=\frac{1}{4}\sin\left(1+x^4\right)+c$

11 $f(x)=\dfrac{\sin x}{2+\cos x}$; let $u=2+\cos x\ \therefore\dots du\equiv\dots-\sin x\,dx$; $\int\dfrac{\sin x}{2+\cos x}dx\equiv-\int\dfrac{1}{u}du=-\ln(2+\cos x)+c$

12 $f(x)=\dfrac{\ln x}{x}=\ln x\left(\dfrac{1}{x}\right)$; let $u=\ln x\ \therefore\dots du\equiv\dots\dfrac{1}{x}dx$; $\int\dfrac{\ln x}{x}dx\equiv\int u\,du=\frac{1}{2}u^2+c=\frac{1}{2}(\ln x)^2+c$

13 $f(x)=\sec^2(4x)$; let $u=4x\ \therefore\dots du\equiv\dots 4dx$; $\int\sec^2(4x)dx\equiv\frac{1}{4}\int\sec^2 u\,du=\frac{1}{4}\tan u+c=\frac{1}{4}\tan(4x)+c$

14 $f(x)=xe^{\frac{1}{4}x}$; let $g(x)=x\ \therefore g'(x)=1$, let $h'(x)=e^{\frac{1}{4}x}\ \therefore h(x)=4e^{\frac{1}{4}x}$;

$$\int xe^{\frac{1}{4}x}dx=4xe^{\frac{1}{4}x}-\int 4e^{\frac{1}{4}x}dx=4xe^{\frac{1}{4}x}-16e^{\frac{1}{4}x}+c=4e^{\frac{1}{4}x}(x-4)+c$$

15 $f(x)=x^2\ln(3x)$; let $g(x)=\ln(3x)\ \therefore g'(x)=\dfrac{1}{x}$, let $h'(x)=x^2\ \therefore h(x)=\dfrac{x^3}{3}$;

$$\int x^2\ln(3x)dx=\tfrac{1}{3}x^3\ln(3x)-\int\frac{x^2}{3}dx=\tfrac{1}{3}x^3\ln(3x)-\tfrac{1}{9}x^3+c=\tfrac{1}{9}x^3(3\ln(3x)-1)+c.$$

16 $f(x)=x\cos(4x)$; let $g(x)=x\ \therefore g'(x)=1$, let $h'(x)=\cos(4x)\ \therefore h(x)=\frac{1}{4}\sin(4x)$;

$$\int x\cos(4x)dx=\tfrac{1}{4}x\sin(4x)-\int\tfrac{1}{4}\sin(4x)dx=\tfrac{1}{4}x\sin(4x)+\tfrac{1}{16}\cos(4x)+c=\tfrac{1}{16}\left(4x\sin(4x)+\cos(4x)\right)+c.$$

17 $f(x)=\dfrac{x}{1+x^4}$ between 0 and 1; let $u=x^2\ \therefore\dots du\equiv\dots 2x\,dx$; $u(0)=0$, $u(1)=1$;

$$\int_0^1\frac{x}{1+x^4}dx=\tfrac{1}{2}\int_0^1\frac{1}{1+u^2}du=\left[\tfrac{1}{2}\tan^{-1}u\right]_0^1=\tfrac{1}{2}\tan^{-1}(1)-\tfrac{1}{2}\tan^{-1}(0)=\tfrac{1}{2}\tan^{-1}(1)=\tfrac{1}{2}\times\tfrac{\pi}{4}=\tfrac{\pi}{8}\approx 0.3927\ \text{(4dp)}$$

18 $f(x)=\dfrac{\cos x}{\sqrt{1-\frac{1}{9}\sin^2 x}}$ between 0 and $\frac{\pi}{2}$; let $u=\frac{1}{3}\sin x\ \therefore\dots du\equiv\dots\frac{1}{3}\cos x\,dx$; $u(0)=0$, $u\left(\frac{\pi}{2}\right)=\frac{1}{3}$;

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{1-\frac{1}{9}\sin^2 x}}dx=3\int_0^{\frac{1}{3}}\frac{1}{\sqrt{1-u^2}}du=\left[3\sin^{-1}u\right]_0^{\frac{1}{3}}=3\sin^{-1}\left(\tfrac{1}{3}\right)-3\sin^{-1}(0)=3\sin^{-1}\left(\tfrac{1}{3}\right)\approx 1.0195\ \text{(4 dp)}$$

19 $f(x)=x\sin\left(\frac{1}{2}x\right)$ between 0 and $\frac{\pi}{3}$; let $g(x)=x\ \therefore g'(x)=1$; let $h'(x)=\sin\left(\frac{1}{2}x\right)\therefore h(x)=-2\cos\frac{1}{2}x$;

$$\int_0^{\frac{\pi}{3}}x\sin\left(\tfrac{1}{2}x\right)dx=\left[-2x\cos\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{3}}+\int_0^{\frac{\pi}{3}}2\cos\left(\tfrac{1}{2}x\right)dx=\left[-2x\cos\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{3}}+\left[4\sin\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{3}}=\left[4\sin\left(\tfrac{1}{2}x\right)-2x\cos\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{3}}$$

$$=\left(4\sin\left(\tfrac{\pi}{6}\right)-\tfrac{2\pi}{3}\cos\left(\tfrac{\pi}{6}\right)\right)-\left(4\sin(0)-2(0)\cos\left(\tfrac{\pi}{6}\right)\right)=4\times\tfrac{1}{2}-\tfrac{2\pi}{3}\times\tfrac{\sqrt{3}}{2}=2-\tfrac{\pi}{3}\sqrt{3}\approx 0.1862\ \text{(4dp)}$$

20 $f(x)=\sqrt{x^3}\ln x$ between 1 and 4; let $g(x)=\ln(x)\ \therefore g'(x)=\frac{1}{x}$; let $h'(x)=x^{\frac{3}{2}}\ \therefore h(x)=\frac{2}{5}x^{\frac{5}{2}}$;

$$\int_1^4 x^{\frac{3}{2}}\ln x\,dx=\left[\tfrac{2}{5}x^{\frac{5}{2}}\ln x\right]_1^4-\int_1^4\tfrac{2}{5}x^{\frac{3}{2}}dx=\left[\tfrac{2}{5}x^{\frac{5}{2}}\ln x\right]_1^4-\left[\tfrac{4}{25}x^{\frac{5}{2}}\right]_1^4=\left[\tfrac{2}{5}x^{\frac{5}{2}}\ln x-\tfrac{4}{25}x^{\frac{5}{2}}\right]_1^4$$

$$=\left(\tfrac{2}{5}(4)^{\frac{5}{2}}\ln(4)-\tfrac{4}{25}(4)^{\frac{5}{2}}\right)-\left(\tfrac{2}{5}(1)^{\frac{5}{2}}\ln(1)-\tfrac{4}{25}(1)^{\frac{5}{2}}\right)=\tfrac{64}{5}\ln 4-\tfrac{128}{25}+\tfrac{4}{25}=\tfrac{128}{5}\ln 2-\tfrac{124}{25}\approx 12.7846\ \text{(4 dp)}$$