Differentiation

The main rules of differentiation are:

$f(x)$	ax^n	e^{ax}	$\ln ax$	$\sin ax$	$\cos ax$	Product: $g(x)h(x)$	Quotient: $\dfrac{g(x)}{h(x)}$	Composite: $g(h(x))$
$f'(x)$	anx^{n-1}	ae^{x}	$\dfrac{1}{x}$	$a\cos ax$	$a\cos ax$	$g(x)h'(x)+g'(x)h(x)$	$\dfrac{h(x)g'(x)-g(x)h'(x)}{(h(x))^2}$	$g'(h(x))h'(x)$

The uses of differentiation are: to determine the gradient of a curve at any point, to find the equation of a tangent to a curve at a specified point, to find the stationary points of a (cubic) function and to solve problems involving velocity and aceleration.

[Rules of Indices (reminder): $x^a \times x^b = x^{a+b}$, $x^a \div x^b = x^{a-b}$, $(x^a)^b = x^{ab}$, $\sqrt[b]{x^a} = x^{\frac{a}{b}}$, $\dfrac{1}{x^a} = x^{-a}$, $x^0 = 1$]

Examples

(1) Find the equation of the tangent to the graph of $y = f(x)$ at the point (1,3), where $f(x) = 8x^2 - 3x - 2$

$f'(x) = 16x - 3 \quad \therefore f'(1) = 13 = m$. Substituting in $y = mx + c$, $3 = 13\times 1 + c \therefore c = -10 \quad \therefore y = 13x - 10$

(2) Find the coordinates of the stationary points of $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - 5$ and sketch the graph of $y = f(x)$

$f'(x) = x^2 + x - 6$; for stationary points $f'(x) = 0 \therefore x^2 + x - 6 = 0 \therefore (x+3)(x-2) = 0 \therefore x = -3$ or $x = 2$

$f(-3) = \frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 - 6(-3) - 5 = \frac{17}{2} \Rightarrow$ P(-3,8.5) $\quad f(2) = \frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 - 6(2) - 5 = -\frac{37}{3} \Rightarrow$ Q(2,-12.$\dot{3}$)

With stationary points P and Q and the intercept *f*(0) = -5, the graph of $y = f(x)$ can be sketched. (not shown)

(3) The height above ground level of a ball that is thrown vertically upwards is given by $s = 15t - 5t^2$.
Find: (a) its initial velocity (b) its maximum height (c) its acceleration (d) how long it takes to hit the ground

(a) $v = \dfrac{ds}{dt} = 15 - 10t$, $\therefore$ when t = 0, v = 15 ms^{-1} (b) when v = 0, t = 1.5 $\therefore$ s = 15(1.5) - 5(1.5)^2 = 11.25 m

(c) $a = \dfrac{dv}{dt} = -10$ ms^{-2} = g (d) when s = 0, $15t - 5t^2 = 0 \therefore t = 0$ or $t = 3$, $\therefore$ it takes 3 seconds to hit the ground.

(4) Using the rules of differentiation above, find the value of the derivative when x = 1 of each of the following functions:

(a) $f(x) = x^5\sqrt{x} = x^5 \times x^{\frac{1}{2}} = x^{\frac{11}{2}} \quad \therefore f'(x) = \frac{11}{2}\times x^{\frac{11}{2}-1} = \frac{11}{2}\times x^{\frac{9}{2}} = \frac{11}{2}\times x^4 \times x^{\frac{1}{2}} = \frac{11}{2}x^4\sqrt{x} \quad \therefore f'(1) = 5.5$

(b) $f(x) = 4e^{-2x} + \ln(7x) \quad \therefore f'(x) = -8e^{-2x} + \dfrac{1}{x} \quad \therefore f'(1) = -8e^{-2} = -1.08$ (to 3 significant figures)

(c) $f(x) = x^3\cos(2x)$; Let $g(x) = x^3 \therefore g'(x) = 3x^2$, Let $h(x) = \cos(2x) \therefore h'(x) = -2\sin(2x)$

$f'(x) = x^3 \times -2\sin(2x) + 3x^2 \times \cos(2x) = x^2(-2x\sin(2x) + 3\cos(2x)) \quad \therefore f'(1) = -2\sin(2) + 3\cos(2) = -3.07$ (3sf)

(d) $f(x) = \dfrac{\cos(x)}{1+\ln(x)}$; Let $g(x) = \cos(x) \therefore g'(x) = -\sin(x)$, Let $h(x) = 1 + \ln(x) \therefore h'(x) = \dfrac{1}{x}$

$$f'(x) = \frac{(1+\ln x)\times -\sin x - \cos x \times \frac{1}{x}}{(1+\ln x)^2} = -\frac{(1+\ln x)\sin x + \frac{1}{x}\cos x}{(1+\ln x)^2} \qquad f'(1) = -(\sin(1)+\cos(1)) = 1.38 \text{ (3 sig fig)}$$

(e) $f(x) = \sqrt{\cos(3x)} = (\cos(3x))^{\frac{1}{2}}$; Let $g(x) = x^{\frac{1}{2}} \therefore g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$; Let $h(x) = \cos(3x) \therefore h'(x) = -3\sin(3x)$

$$f'(x) = \tfrac{1}{2}(\cos(3x))^{-\frac{1}{2}} \times -3\sin(3x) = -\frac{3\sin(3x)}{2\sqrt{\cos(3x)}}$$

Exercises

(1) Find the equation of the tangent to the graph of $f(x) = \frac{1}{2}x^4 - x^3 + 2x + 6$ at the point (2,10).

(2) Find the coordinates of the stationary points of $f(x) = 2x^3 - 15x^2 + 24x + 20$ and sketch the graph of $y = f(x)$

(3) The distance of an object moving in a straight line from a fixed starting point is given by $s = -\frac{1}{2}t^2 + 6t + 1$ m.
Find: (a) initial velocity (b) distance when object is at rest (c) velocity when object returns to its starting point.

(4) Find the value of the derivative when x = 2 of each of the following functions, giving your answers to 3 sig fig.

(a) $f(x) = \sqrt[5]{x^8}$ (b) $f(x) = x^3e^{-x}$ (c) $f(x) = \dfrac{x^2 + x}{\cos x}$ (d) $f(x) = \cos(\ln(2x))$

Answers (1) $y = 6x - 2$ (2) P(1,31), Q(4,4) (3)(a) 6ms^{-1} (b) 19m (c) -6.16ms^{-1} (4)(a) 2.43 (b) 0.541 (c) 19.5 (d) -0.492