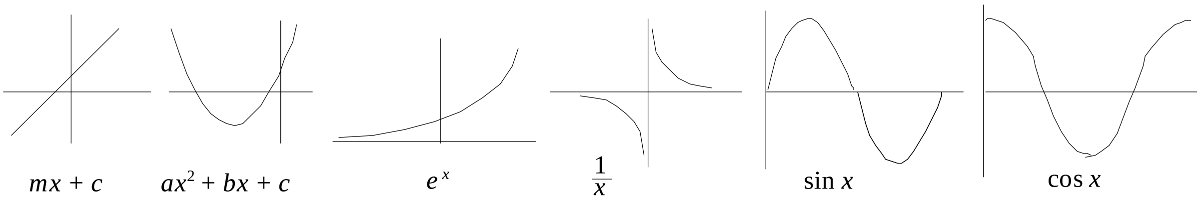
Functions, Inverses and Graph Sketching

A function can be understood as a <u>process</u> with inputs and outputs. For each output a 'one-to-one' function has one input, a 'many-to-one' function has two or more inputs. A function is specified by a description of its process and of its domain (which is the set of its inputs) eg: $f(x) = 3x + 2,\ x \in \mathbb{R}$. Sketches like those below help to distinguish between 'one-to-one' and 'many-to-one' functions.



A line parallel to the *x*-axis will cross the graph of a one-to-one function once only.

Inverse functions
To find the inverse of a given function, substitute *y* for *f(x)*, make *x* the subject, interchange *x* and *y*. Only a one-to-one function can have an inverse so in the above selection the linear, exponential and rational functions have inverses. Quadratic and trigonometric functions do not have inverses, but 'many-to-one' functions can be made 'one-to-one' if the domain is suitably restricted.

Example 1: $f(x) = 3x + 2,\ x \in \mathbb{R}$; inverse: $y = 3x + 2 \quad \therefore x = \frac{1}{3}(y-2) \quad \therefore f^{-1}(x) = \frac{1}{3}(x-2),\ x \in \mathbb{R}$ as shown in Figure 1 below. Notice that the inverse is a reflection of the function in the line $y = x$

Example 2: $f(x) = x^2 - 4x - 2,\ x \in \mathbb{R}$; from the 'completed-square' form: $f(x) = (x-2)^2 - 6,\ x \in \mathbb{R}$, we deduce that: $x \geq 2$ for an inverse to exist, and $f(2) = 6$, so: $f(x) = x^2 - 4x - 2,\ x \in \mathbb{R},\ x \geq 2$ can be drawn as shown in Figure 2 below, with an inverse function: $f'(x) = \sqrt{x+6} + 2, \quad x \in \mathbb{R}, \quad x \geq -6$.

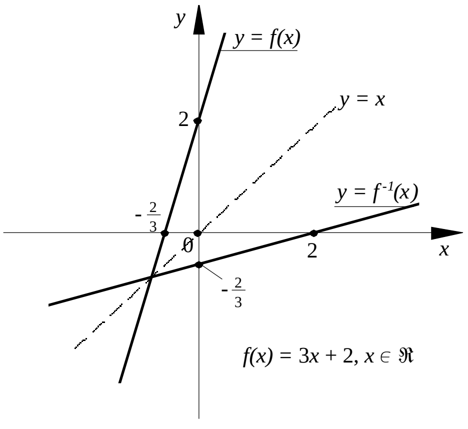


Figure 1

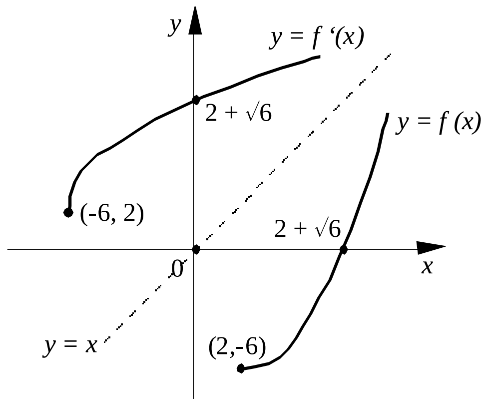


Figure 2

Exercises

For each of the following, sketch the graph and its inverse on the same axes, labelling both graphs clearly with complete function descriptions and showing all intercepts and end-points.

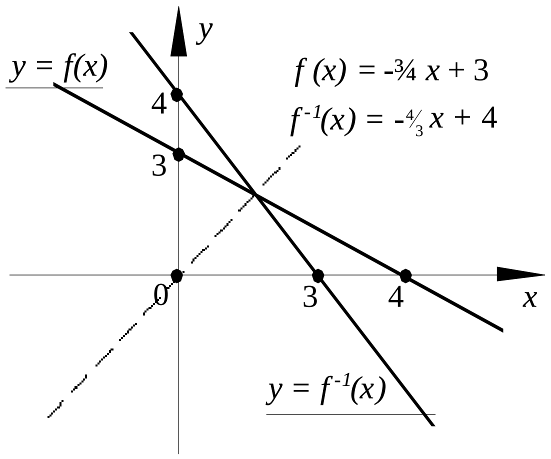
1 $f(x) = -\frac{3}{4}x + 3,\ x \in \mathbb{R}$ **2** $f(x) = 2x - 1,\ x \in \mathbb{R},\ x$ in [-1, 1] **3** $f(x) = e^{x+1},\ x \in \mathbb{R}$

For each of the following, sketch the graph of the function with a restricted domain to allow for an inverse function, show end-points and intercepts on the co-ordinate axes, and define the image set.

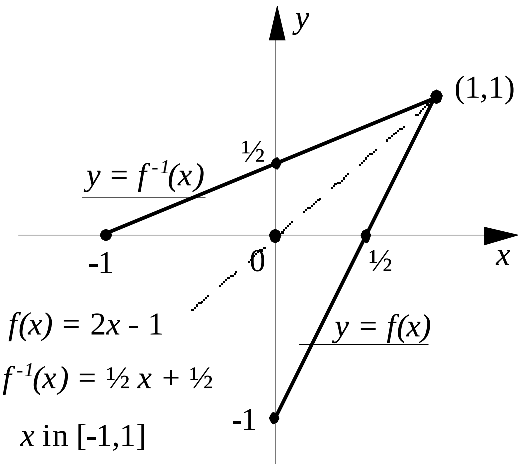
4 $f(x) = x^2 + 4x + 2,\ x \in \mathbb{R}$ **5** $f(x) = x^2 - 4x + 4,\ x \in \mathbb{R}$ **6** $f(x) = 2x^2 + 4x - 2,\ x \in \mathbb{R}$

Solutions to Exercises

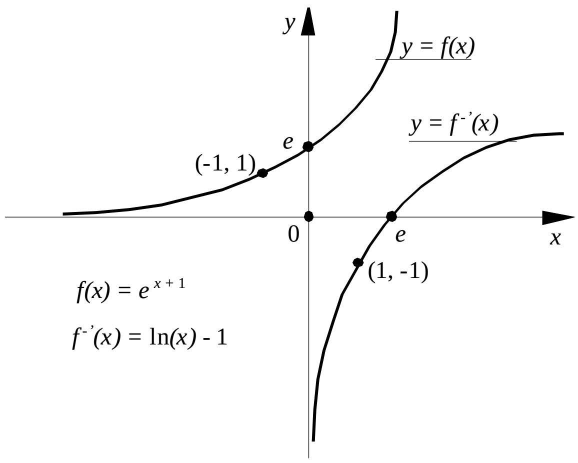
Question 1



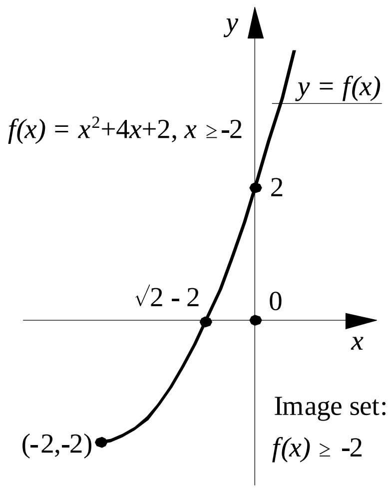
Question 2



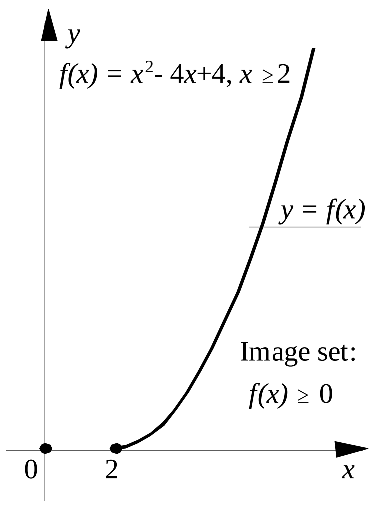
Question 3



Question 4



Question 5



Question 6