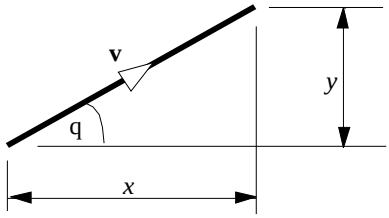
Vectors

The matrix $V = \begin{pmatrix} x \\ y \end{pmatrix}$ is a column vector which is written as $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$

where **i** is a horizontal and **j** is a vertical component of unit length.

The vector **v** has magnitude $|\mathbf{v}| = \sqrt{x^2 + y^2}$ and direction $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

To find **v** given magnitude and direction: $\mathbf{v} = (|\mathbf{v}|\cos\theta)\mathbf{i} + (|\mathbf{v}|\sin\theta)\mathbf{j}$



Addition of Vectors

Addition of vectors links them end-to-end in directional sequence. For example:

Given: $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{c} = -4\mathbf{i} + 2\mathbf{j}$

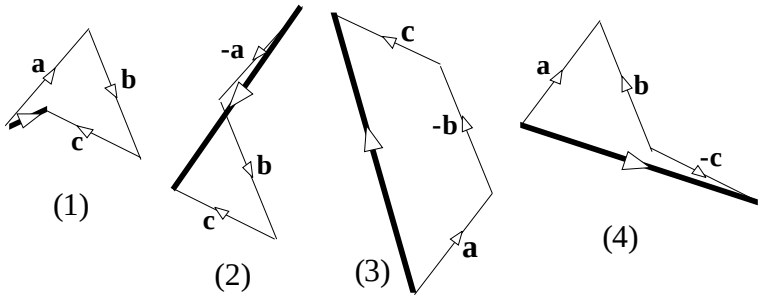
then: (1) $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{i} + \mathbf{j}$

(2) $-\mathbf{a} + \mathbf{b} + \mathbf{c} = -5\mathbf{i} - 7\mathbf{j}$

(3) $\mathbf{a} - \mathbf{b} + \mathbf{c} = -3\mathbf{i} + 11\mathbf{j}$

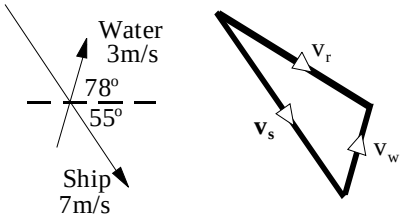
(4) $\mathbf{a} + \mathbf{b} - \mathbf{c} = 9\mathbf{i} - 3\mathbf{j}$

(Scalar multiples of vectors are calculated as for matrices, thus: $2\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$, etc.)



Example of using vector components

In still water, the speed of a ship is 7ms^{-1} with bearing S35°E. Use a triangle of velocities to find its speed and bearing when it is sailing against a current of 3ms^{-1} flowing from the direction S12°W.



We want the resultant of the given velocities.

$\mathbf{v}_s = (7\cos 55^\circ)\mathbf{i} - (7\sin 55^\circ)\mathbf{j} \quad \therefore \mathbf{v}_s = 4.02\mathbf{i} - 5.73\mathbf{j}$

$\mathbf{v}_w = (3\cos 78^\circ)\mathbf{i} + (3\sin 78^\circ)\mathbf{j} \quad \therefore \mathbf{v}_w = 0.62\mathbf{i} + 2.93\mathbf{j}$

$\mathbf{v}_r = \mathbf{v}_s + \mathbf{v}_w = 4.64\mathbf{i} - 2.80\mathbf{j} \quad \therefore |\mathbf{v}_r| = \sqrt{4.64^2 + 2.80^2} = \mathbf{5.42ms^{-1}}$

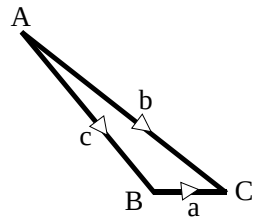
For bearing of $\mathbf{v}_r$, $\theta = \tan^{-1}\left(\frac{-2.80}{4.64}\right) = 31.1^\circ$, therefore bearing is: **S58.9°E**

Triangle of vectors

In some cases it is simpler to use the Sine and Cosine Rules to solve a triangle of vectors. Using the standard notation for triangles, the Cosine Rule is: $a^2 = b^2 + c^2 - 2bc\cos A$ and the Sine Rule is: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Example of using a triangle of vectors

A boat has a speed in still water of 10ms^{-1} and is pointed south-east, but there is a current of 3ms^{-1} flowing east. Find the resultant speed of the boat, in terms of speed and bearing, to one decimal place.



Let **a** = speed of current = 3 at an angle of 0° (east); Let **b** = velocity of boat;
Let **c** = speed of boat in still water = 10 at an angle of -45° (south-east)
In the triangle of vectors, therefore angle B = 135°

Using the Cosine Rule: $\mathbf{b}^2 = 3^2 + 10^2 - 2\times 3\times 10\times\cos 135^\circ = 151.4 \quad \therefore \mathbf{b} = 12.3$

Using the Sine Rule: $\frac{3}{\sin A} = \frac{12.3}{\sin 135^\circ} \quad \therefore A = \sin^{-1}\left(\frac{\sin 135^\circ}{12.3}\right) = 9.9^\circ$

Angle of **b** from vertical = 9.9° + 45° ∴ velocity of boat = **12.3ms^{-1} at S54.9°E**

Exercises

1) A boat has a speed in still water of 4ms^{-1} in the direction S10°E but there is a current of speed 2.5ms^{-1} flowing towards the direction N75°W. Use vector components to find the resultant speed of the boat and its bearing.

2) A bird has a speed in still air of 15ms^{-1} flying in the direction S80°W, but it flies in a wind of speed 8ms^{-1} blowing from S5°E. Use vector components to find the velocity of the bird relative to the ground.

3) An object at rest is acted upon by three forces: **p**, **q** and **r**. If **p** has magnitude 130N and direction 58° and **q** has magnitude 85N and direction -27°, use a triangle of forces to find the magnitude and direction of **r**.

4) An object of mass 20kg is hanging from a horizontal ceiling on two cords. One cord makes an angle of 50° with the ceiling and the tension in the cord is 187.7N. Use a triangle of forces to find the tension in the other cord and the angle it makes with the ceiling. (Assume that g = -10 ms^{-2})

Answers 1) 3.7ms^{-1}, S28°W 2) 16.4ms^{-1}, N70.9°W 3)161.4N, -153.6° 4) 133.1N, 25°