Chapter 1: Proof by Induction

1) You are probably familiar with these formulas for the sum of series:

$\sum_{s=1}^{n} s = \frac{1}{2}n(n+1)$ is the sum of the numbers 1 to *n*.

$\sum_{s=1}^{n} s^2 = \frac{1}{6}n(n+1)(2n+1)$ is the sum of the squares of the numbers 1 to *n*.

$\sum_{s=0}^{n-1} r^s = \frac{1-r^n}{1-r}$ is the sum of the geometric series $u^n = r^{n-1}$, n = 1, 2, 3 ...

You have probably known a proof of some or all of these before. See if you can write a proof of each of these formulas now.

The proofs you chose for the formulas above were probably "constructive proofs". These proofs construct the formula from basic principles and prove it as valid. There is another method of proving formulas that seems almost like a magic trick because it can prove whether a formula is true without knowing, or telling, anything about how the formula was constructed.

2) Look at the formula for $\sum_{s=1}^{n} s = \frac{1}{2}n(n+1)$. Does it work when n = 1? When n = 2? When n =3?

Now let us <u>assume</u> that it works for all numbers up to some number k. (k = 1 will do.)
Does that mean it also works for k+1?

In other words if $\sum_{s=1}^{k} s = \frac{1}{2}k(k+1)$ does it then follow that $\sum_{s=1}^{k+1} s = \frac{1}{2}k(k+1)(k+2)$?

(Hint: $\sum_{s=1}^{k+1} s = \sum_{s=1}^{k} s + (k+1)$.)

If you can answer "Yes" to the question above you have proved that the formula is true for n = 1 and if it is true for n = 1 then it is also true for n = 2. And if it is true for n = 2 then it is also true for n = 3, and so on. So it must be true for any value of n.

The slightly unsatisfactory property of the method is the fact that it gives no clue on where the formula came from. You have to come up extreme generality of the method. It provides relatively simple proofs for a very wide range of formulas.

3) Use the method of mathematical induction to prove the formulas for the quadratic series $\sum_{s=1}^{n} s^2 = \frac{1}{6}n(n+1)(2n+1)$.

Since this needs to be a formal proof you should use a precise format and set of words for your answer. Follow the format below in every proof by induction that you do. Gaps in the proof, shown as "_________", are there for you to fill in yourself.

Let $S(n) = \sum_{s=1}^{n} s^2$ and $T(n) = \frac{1}{6}(n+1)(2n+1)$.

We will prove by induction the formula that S(n) = T(n) for all integers n..

$$S(1)=\sum_{s=1}^{1} s^2 = \_\_\_\_\_\_\_\_ = \frac{1}{6}(1)(1+1)(2(1)+1) = T(1)$$

So S(1) = T(1) and the formula is true when n = 1.
We will assume that S(n) = T(n) for all values of n up to a general value k.
In that case,

$$S(k+1)=\sum_{s=1}^{k+1} s^2 = \sum_{s=1}^{k} s^2 + \_\_\_\_\_\_\_\_ = \frac{1}{6}k(k+1)(2k+1) + \_\_\_\_\_\_\_\_ \quad \text{(by assumption)}$$

$$= \_\_\_\_\_\_\_\_\_\_\_ = T(k+1)$$

We have shown that the formula is true for n= 1, and also that if it is true for all n up to a general number k then it is also true for n = k+1. Therefore, by the principle of mathematical induction, the formula is true for all positive integers.

This is the form of words that you should use whenever proving a statement by mathematical induction.

4) Use the same method, and the same format, to prove that $\sum_{s=0}^{n-1} r^s = \frac{1-r^n}{1-r}$.

5) Show by mathematical induction that $3^{2n}+11$ is divisible by 4.

6) Research the methods of “modular arithmetic”. Use this to prove the above theorem more directly.

A recurrence relation is a way of specifying an infinite sequence by giving the first term or terms and specifying a rule that shows how the next term is defined in terms of the preceding terms.
For example : $u_1 = 1$, $u_{n+1} = 3u_n + 4$ is a recurrence relation.

7) Prove by mathematical induction that the recurrence relation $u_1 = 1$, $u_{n+1} = 3u_n + 4$ is equivalent to the formula $u_n = 3^n - 2$.

8) Prove by mathematical induction that $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$ for all integers n.