Chapter 10: Further Calculus

1) Find the integrals of i) $\frac{1}{3x+1}$ and ii) $\frac{2}{x-1}+\frac{1}{3x+1}$.

2) Show that $\frac{2}{x-1}+\frac{1}{3x+1}=\frac{7x+1}{(x-1)(3x+1)}$ and hence show that

$\int \frac{7x+1}{(x-1)(3x+1)}dx = 2\ln(x-1)+\frac{1}{3}\ln(3x+1)$.

We can find the integral in Question 2 but it depended on us knowing from Question 1 that $\frac{7x+1}{(x-1)(3x+1)}=\frac{2}{x-1}+\frac{1}{3x+1}$. If we are to do this again we need a method to separate the single fraction into two simpler fractions as shown. We do this by making use of the fact that the fraction $\frac{ax+b}{(cx+d)(ex+f)}$ can *always* be written as the sum of two simpler fractions $\frac{\mathrm{P}}{cx+d}+\frac{\mathrm{Q}}{ex+f}$ (which are called "*partial fractions*".)

3 i) Show that if $\frac{ax+b}{(cx+d)(ex+f)}=\frac{\mathrm{P}}{cx+d}+\frac{\mathrm{Q}}{ex+f}$ then by multiplying through by the left hand denominator we get $ax+b=\mathrm{P}(ex+f)+\mathrm{Q}(cx+d)$.

ii) By matching coefficients of x on both sides, show that we get two simultaneous linear equations, which can be written in matrix form as $\begin{pmatrix} e & c \\ f & d \end{pmatrix}\begin{pmatrix} \mathrm{P} \\ \mathrm{Q} \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}$.

iii) Show that this will have a solution for P and Q unless $ed=cf$, in which case the denominator of the left hand fraction will be of the form $(cx+d)^2$.

iii) Show that the solution is given by $\begin{pmatrix} \mathrm{P} \\ \mathrm{Q} \end{pmatrix}=\begin{pmatrix} e & c \\ f & d \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$.

It is not required to solve the simultaneous equations by matrices and usually they will be simple enough to solve with direct algebra. But even then there is another method that is often simpler still. The equation $ax+b=\mathrm{P}(ex+f)+\mathrm{Q}(cx+d)$ is an *identity*, which means it must be true *whatever the value of* x , which means you can choose values of x which give simple equations.

4 i) Show that the equation $\frac{7x+1}{(x-1)(3x+1)}\equiv\frac{\mathrm{P}}{x-1}+\frac{\mathrm{Q}}{3x+1}$ requires that

$7x+1\equiv \mathrm{P}(3x+1)+\mathrm{Q}(x-1)$.

ii) Substitute $x = 1$ into the identity above to get a simple equation to find P ; then substitute $x = -\frac{1}{3}$ to get a simple equation to find Q and hence write $\dfrac{7x+1}{(x-1)(3x+1)}$ as a sum of partial fractions.

iv) Hence show that $\displaystyle\int \frac{7x+1}{(x-1)(3x+1)}\,dx = 2\ln(x-1)+\tfrac{1}{3}\ln(3x+1)$.

5) In Question 3 ii) we showed that the form of partial fractions in Question 4 will not work for a fraction of the form $\dfrac{ax+b}{(cx+d)^2}$.

i) Show that $\dfrac{ax+b}{(cx+d)^2}$ can be written as $\dfrac{\text{P}}{(cx+d)^2}+\dfrac{\text{Q}}{cx+d}$ and find P and Q in terms of a, c and d.

ii) Show that $\dfrac{2x+1}{(x-4)^2}$ can be written as $\dfrac{\text{P}}{(x-4)^2}+\dfrac{\text{Q}}{x-4}$, giving the values of P and Q.

iii) Hence find $\displaystyle\int \frac{2x+1}{(x-4)^2}\,dx$.

6) Imagine you had invented two functions called $s(x)$ and $c(x)$ and all you knew about these functions were these properties: $s(0)=0, c(0)=1, \dfrac{d}{dx}\big(s(x)\big)=c(x)$ and $\dfrac{d}{dx}\big(c(x)\big)=-s(x)$.

i) Show that $\dfrac{d}{dx}\big(s^2(x)\big)+\big(c^2(x)\big)=0$ and so therefore $s^2(x)+c^2(x)$ is a constant.

ii) Use the initial conditions $s(0)=0, c(0)=1$ to show that $s^2(x)+c^2(x)\equiv 1$.

iii) Show that $s(x)=\sqrt{1-c^2(x)}$ and $c(x)=\sqrt{1-s^2(x)}$.

Assume that $c(x)$ and $s(x)$ have inverses $c^{-1}(x)$ and $s^{-1}(x)$ and we want to find the derivatives of these functions. Follow this argument:

If $y=s^{-1}(x)$ then $x=s(y)$ and so $\dfrac{dx}{dy}=c(y)=\sqrt{1-s^2(y)}=\sqrt{1-x^2}$; but $\dfrac{dx}{dy}=\dfrac{1}{\frac{dy}{dx}}$ so the derivative of $s^{-1}(x)$ is $\dfrac{1}{\sqrt{1-x^2}}$.

7 i) Repeat a similar argument as that above to show that the derivative of is $c^{-1}(x)$ is $\dfrac{-1}{\sqrt{1-x^2}}$.

Of course you have realised by now that $c(x)$ and $s(x)$ are the familiar functions $\cos(x)$ and $\sin(x)$. We used them previously in the chapter on Maclaurin series. I wanted to separate them from their normal names to demonstrate that their properties do not have to depend on their geometrical function in terms of angles. They arise directly from a few small rules that can be defined independently of any context. This is what mathematicians call *abstraction.* It means isolating the essential properties from any extraneous details.

8) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

i) Show that $1 + \tan^2(x) = \dfrac{1}{\cos^2(x)} \equiv \sec^2(x)$.

ii) Show that $\dfrac{d}{dx}\tan(x) = \dfrac{1}{\cos^2(x)} \equiv \sec^2(x)$.

Hence show that $y = \tan(x)$ is a solution to the differential equation $\dfrac{dy}{dx} = 1 + y^2$.

iii) By using a similar argument as in question 7, show that the derivative of $\tan^{-1}(x)$ is $\dfrac{1}{1+x^2}$.

The functions $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are called, respectively, $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$. There is a reason for this name.

9 i) Show that the curve with parametric form $x = \cos(t)$ and $y = \sin(t)$ is the unit circle with the Cartesian form $x^2 + y^2 = 1$.

(This is why sine and cosine are sometimes called the “circular” functions.)

ii) Draw the unit circle on the x-y axes and draw a radius to a point on the circle in the first quadrant, with co-ordinates (x,y). Call the angle of the radius with the horizontal θ. Show that the length of the arc is θ and that $\theta = \sin^{-1}(y)$ and $\theta = \cos^{-1}(x)$. So the length of the arc between the point (x,y) and the x-axis is $\arcsin(y)$ and also $\arccos(x)$.

We know that the derivative of $\arcsin(x)$ is $\dfrac{1}{\sqrt{1-x^2}}$ and that the derivative of $\arccos(x)$ is $-\dfrac{1}{\sqrt{1-x^2}}$ and that the derivative of $\arctan(x)$ is $\dfrac{1}{1+x^2}$. Every time you get a derivative you get an integral in reverse. This should suggest that we have discovered some more integralsd we can use .

10 i) If the *derivative* of $\arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$ show that the *integral* of $\frac{1}{\sqrt{1-x^2}}$ is $\arcsin(x)+C$.

And since $\arcsin(0)=0, C=0$ and so the integral of $\frac{1}{\sqrt{1-x^2}}$ is $\arcsin(x)$.

ii) If the derivative of $\arccos(x)$ is $\frac{-1}{\sqrt{1-x^2}}$, show that the integral of $\frac{-1}{\sqrt{1-x^2}}$ is $\arccos(x)+C$.

And since $\arccos(0)=\frac{\pi}{2}, C=\frac{\pi}{2}$ and the integral of $\frac{1}{\sqrt{1-x^2}}$ is also equal to $-\arccos(x)+\frac{\pi}{2}$.

iii) Conclude that $\arcsin(x)=-\arccos(x)+\frac{\pi}{2}$.

iv) Sketch a plot of $\arcsin(x)$ and $\arccos(x)$ on the same axes so as to demonstrate the statement in (iii).

11) Demonstrate by two different methods that the integral of $\frac{1}{1+x^2}$ is $\arctan(x)$.

i) First start with the derivative of $\arctan(x)$ and use the initial condition $\arctan(0)=0$.

ii) Then make the substitution $u=\tan(x)$ in $\int \frac{1}{1+x^2}dx$.

12 i) Show that $1-x^2+x^4-x^6+x^8-\cdots=\frac{1}{1+x^2}$.

ii) Hence show that $\int\left(1-x^2+x^4-x^6+x^8-\cdots\right)dx=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\cdots$.

iii) Hence show that $\tan^{-1}(x)=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\cdots$.

iv) Hence write down a very lovely series for $\frac{\pi}{4}$ that uses all the odd numbers in order.

This series has some history behind it. It is usually called “Leibniz’ Series” after Gottfried Leibniz, (1646 - 1716) who is credited, jointly with Sir Isaac Newton, of inventing the calculus. However the series also known as “Gregory’s Series” after James Gregory (1638 - 1675) who seems to have discovered it some years before Leibniz published. However, it also seems to have been known much earlier by the Indian mathematician Sangamagrama Madhava (c. 1350 – c. 1425).
He described a method of repeated calculation which would produce this formula $\theta=\tan(\theta)-\tfrac{1}{3}\tan^3(\theta)+\tfrac{1}{5}\tan^5(\theta)-\tfrac{1}{7}\tan^7(\theta)+\cdots$, the same formula published by James Gregory more than two centuries later.

13) Use the formula above to produce a series for $\frac{\pi}{4}$.

I hope your instinct was to reach for a calculator and calculate enough terms of the series to show it really does approach $\frac{\pi}{4}$. If so, you may have foubnd it a bit frustrating. This series is elegant, but not efficient. It converges agonisingly slowly and so was not much good as an estimator of π. There are many other series for π. You may wish to explore them some time.

14 i) By making the substitution $x=\sin(u)$, show that $\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x)$.

ii) By making the substitution x = cos(u), show that $\int \frac{1}{\sqrt{1-x^2}} = -\arccos(x)+\frac{\pi}{2}$.

iii) Conclude that $\arcsin(x) = -\arccos(x)+\frac{\pi}{2}$.

15) Find $\int \frac{1}{\sqrt{a^2-x^2}}dx$. (Hint: Write $\frac{1}{\sqrt{a^2-x^2}}$ as $\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}$. Then make the substitution $u=\frac{x}{a}$ and proceed as in question 14.

16 i) Find $\int_1^2 \frac{4}{25+9x^2}dx$. (Hint: Write $\frac{4}{25+9x^2}$ as $\frac{4}{25}\left(\frac{1}{1+\left(\frac{3x}{5}\right)^2}\right)$. Then make the substitution $u=\frac{3x}{5}$ and proceed as in question 10.)

ii) Check your results with a calculator.

17) Find $\int \frac{1}{x^2-8x+20}dx$. (Hint: Complete the square to write $x^2-8x+20$ as $(x-4)^2+4$ and write the integral as $\frac{1}{4}\int \frac{1}{1+\left(\frac{x-4}{2}\right)^2}dx$. Then make the substitution $u=\frac{x-4}{2}$.)