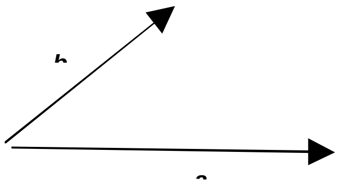
Chapter 11: Vectors

A vector is a one-column matrix. It can also be considered as an arrow. It can also be considered as an extended form of a "number" in its own space with its own form of arithmetic. Vectors have a number of different manifestations and a great number of uses in a wide range of areas. For the moment, let's consider a vector as an arrow. It has a length and a direction.



1 i) Vectors **a** and **b** are shown as left. Copy this diagram and mark the angle between the vectors as θ. Draw a dotted line from the end of vector **b** to meet **a** at a right angle. Mark as a vector the part of vector **a** that finishes at the dotted line and call it **c**.
ii) We call the length of the arrow the "magnitude" of the vector and write the magnitude of vector **a** as |**a**|. Show that the magnitude of the vector **c** is $|\mathbf{b}|\cos(\theta)$.

c is called the "projection" of vector **b** on vector **a**.

iii) Show that **c** may be written as $|\mathbf{b}|\cos(\theta)\hat{\mathbf{a}}$ where $\hat{\mathbf{a}}$ is a unit vector in the direction of **b**. (A "unit vector" is a vector of length 1.)

2 i) Draw a dotted line from the end of vector **a** to meet vector **b** at a right angle . (You might need to add a dotted extension to vector **b**. Then the line you draw will meet *the direction of* **b**.)
ii) Mark as a vector the extension of vector **b** that finishes at the dotted line and call it **d**.
iii) Show that the magnitude of vector **d** is $|\mathbf{a}|\cos(\theta)$.

d is called the "projection" of vector **a** on vector **b**.

iv) Show that **d** may be written as $|\mathbf{a}|\cos(\theta)\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**.

We are going to define a way of multiplying vectors that is similar to the process of projection of one vector on another. We want the *product* of our two vectors to be a *scalar*, which is one-dimensional quantity, like a number. (Vectors are a kind of two-dimensional quantity. Even in three or more dimensions they represent, in some sense, a two-dimensional quantity.) We therefore call this a "*scalar product"* and define it as $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$, where θ is the angle between the two vectors. The scalar product is also known as the "*dot product*".

3) **a** and **b** are two vectors.
i) What is **a.b** when **a** and **b** are parallel?
ii) What is **a.b** when **a** and **b** point in opposite directions?
iii) What is **a.b** when **a** and **b** are perpendicular?
iv) What is **a.a** ?

4) Show that the projection of of vector **a** on vector **b** may be written as $(\mathbf{a}.\hat{\mathbf{b}})\hat{\mathbf{b}}$ and the projection of vector **b** on vector **a** may be written as $(\mathbf{a}.\hat{\mathbf{b}})\hat{\mathbf{a}}$.

5 i) **a** is a vector with modulus (or length) 3 pointing horizontally to the right. Draw vector **a**.
b is a vector with modulus (or length) 2 pointing vertically upwards. Draw vector **b**, with the "tail" of **b** starting at the "nose" of **a**. Draw a third vector that starts at the *tail* of **a** and ends at the *nose* of **b**. Call this vector **c**. Vector **c** will complete the triangle formed of **a**, **b** and **c**.
ii) What you have just done is *vector addition*. Vector **c** is the *sum* of vectors **a** and **b**, so **c = a + b**.
Find the *length* (or modulus) of vector **c** and the *angle* of vector **c**.
iii) **-c** is a vector identical to **c** but pointing in the opposite direction. Show that **a + -c = b** and **b + -c = a**.
This can also be written as **a - c = b** and **b - c = a**.
iv) Find the angle of vector **-c**.

6 i) **i** is a vector with modulus (or length) 1 pointing horizontally to the right. Draw vector **i** . **j** is a vector with modulus (or length) 1 pointing vertically upwards. Draw vector **j** , with the "tail" of **j** starting at the "nose" of **i** .
ii) Show that vector **a** in question 5 equals 3 times vector **i** and vector **b** in question 5 equals 2 times vector **j** .
iii) Show that vector **c** in question 5 equals 3**i** + 2**j** .

i and **j** are called *orthogonal unit vectors* - "orthogonal" because they are perpendicular to each other and "unit" because they have modulus one. **i** always points in the "positive x" direction - horizontally to the right. **j** always points in the "positive y" direction - vertically upwards. We can use **i** and **j** as a basis for all other vectors to create what we call a *vector space.*

7 i) Mark a point on your page as "O" - the "origin". Draw vectors **i** and **j** at the origin.
ii) Draw a vector starting at O of any length in any direction. Convince yourself that this vector (call it **v**) can be written as a sum of multiples of **i** plus multiples of **j .** In other words **v =** p**i +** q**j**, for some real numbers p and q.

8 i) Draw x-y axes and draw points P(1,1) and Q(2,-3) on the plane.
ii) Draw a vector **p** from the origin to the point P and and a vector **q** from the origin to the point Q. **p** and **q** are called the "position vectors" of P and Q.
iii) Show that **p = i + j** and **q =** 2**i -** 3**j** .
iv) Draw the vector **p + q** and show that **p + q =** 3**i -**2**j** .
v) Find from your diagram the angle of vector **p** and the angle of vector **q** and the angle between vector **p** and vector **q** .

9) Show by drawing an arrow diagram that the vector that goes from P to Q is **q - p** and the vector that goes from Q to P is **p** - **q** .

There is another way of expressing the scalar product of **p** and **q.** If $\mathbf{p} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{q} = x_2\mathbf{i} + y_2\mathbf{j}$, then $\mathbf{p.q} = x_1x_2 + y_1y_2$. We will explain a bit later why this is equivalent to the definition $\mathbf{p.q} = |\mathbf{p}||\mathbf{q}|\cos(\theta)$, where θ is the angle between **p** and **q**. But first, you will see that it gives us a quicker way to find the angle between **p** and **q**.

10 i) If $\mathbf{p} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{q} = x_2\mathbf{i} + y_2\mathbf{j}$ show that the angle between **p** and **q** is given by $\cos(\theta) = \dfrac{x_1x_2 + y_1y_2}{|\mathbf{p}||\mathbf{q}|}$.
ii) Find the angle between vectors **p = i + j** and **q =** 2**i -** 3**j** and show that it agrees with your answer in question 8.

With **i** and **j** we have created a two-dimensional vector space, which works well on the page but in a three dimensions we need a three-dimensional vector space. So we need a third orthogonal unit vector, which we call **k**. By tradition we orientate our axes using the *right hand rule*.

11 i) Draw **i** and **j** as unit vectors in the horizontal and vertical direction on the page. Hold the page vertically and place your right hand against vector **i** , fingers pointing in the same direction as **i** . Curl your fingers towards **j** . Which way does your thumb point? That is the direction of **k** .
ii) Draw **i, j** and **k** as three orthogonal unit vectors on the page, using perspective to give a convincing 3D appearance. Place your right hand against vector **i** , fingers pointing in the same direction as **i** . Curl your fingers towards **j** . Your thumb point should point in the direction of **k** .

We would expect the projection of a vector onto *itself* to be its *modulus*. And we would expect the projection of a vector onto an *orthogonal* vector to be zero, because that's what orthogonal means - "at right angles to". Therefore, and since **i , j** and **k** are unit vectors, we would expect the following rules:
i.i = j.j = k.k and **i.j = i.k = j.k =** 0 .

12) Show that if, by definition, $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$, then the rules for **i, j** and **k** are satisfied when this is applied to scalar products of **i, j** and **k**.

13) In a three-dimensional vector space, any vectors **a** and **b** can be defined in terms of the unit vectors, $\mathbf{a} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $\mathbf{b} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$. In that case $\mathbf{a}.\mathbf{b} = (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}).(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})$. Assume here that all the normal rules of arithmetic apply and show that $\mathbf{a}.\mathbf{b} = x_1y_1 + x_2y_2 + x_3y_3$.

Name	Integral equations (SI convention)	Differential equations (SI convention)	Differential equations (Gaussian convention)
Gauss's law	$\oiint_{\partial\Omega} \mathbf{D}\cdot d\mathbf{S} = \iiint_{\Omega} \rho_f \, dV$	$\nabla\cdot\mathbf{D} = \rho_f$	$\nabla\cdot\mathbf{D} = 4\pi\rho_f$
Gauss's law for magnetism	$\oiint_{\partial\Omega} \mathbf{B}\cdot d\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot d\boldsymbol{\ell} = -\frac{d}{dt}\iint_{\Sigma} \mathbf{B}\cdot d\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{H}\cdot d\boldsymbol{\ell} = \iint_{\Sigma} \mathbf{J}_f\cdot d\mathbf{S} + \frac{d}{dt}\iint_{\Sigma} \mathbf{D}\cdot d\mathbf{S}$	$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$	$\nabla\times\mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}\right)$

This is a display (courtesy of Wikipedia https://en.wikipedia.org/wiki/Maxwell%27s_equations) of Maxwell's equations of electromagnetism expressed in vector notation. We won't be considering anything as advanced as this. But it shows how central vector analysis was to the development of laws for electricity, through electromagnetic waves and on to relativity.

We are going to look at the equation of a line in space. You will be quite familiar with the *Cartesian equation* of a line in two dimensions. We are going to see if we can find the equation of a line expressed in terms of *vectors*. At this point it is worth asking the question "What is the difference between a vector and a straight line?"

14) Try and answer these questions: "Does a *line* have a *position*?" "Does a *vector* have a *position*?" "Does a *line* have a *direction*?" "Does a *vector* have a *direction*?" "Does a *line* have a *length*?" "Does a *vector* have a *length*?"

15 i) Write the Cartesian equation of a line with gradient 3 that goes through the point (1,2).
ii) The purpose of the equation of a line is to have a simple method of determining whether or not a point (x,y) lies on the line. Does your equation do this? Given an example.
iii) Sketch your line in question i). Mark point P(1,2) and draw vector $\overrightarrow{\mathbf{OP}}$. Call this vector **a**. Draw another vector, of any length, in the direction of the line, starting at P. Call this vector **b**. Mark a general point R(x,y) anywhere on the line and draw $\overrightarrow{\mathbf{OR}}$. Call this vector **r**. Show that R, and hence any point on the line, can be identified by its position vector $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where **r** is a variable representing a vector and λ is a variable representing a real number.
iv) Show that the point the (3,2) does not lie on the line with that equation but the point (2, 5) does lie on the line.

16 i) Now, in three dimensions, write the vector equation of a straight line that goes through the point P(1,1,1) in the direction of vector **q = i + j - k**.
ii) Test your equation by substituting in the position vector of a point on the line and the position vector of a point not on the line.

17 i) Show that the vector equation of the line may also be written in the form $\mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$.
ii) Let **a, b** and **r** be general 3D vectors so that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = k\mathbf{i} + l\mathbf{j} + m\mathbf{k}$. Substitute **a, b** and **r** into the equation $\mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$ and equate the coefficients of **i**, **j** and **k** to get the connected equations $\frac{x - x_1}{k} = \frac{y - x_2}{l} = \frac{z - x_3}{m} = \lambda$.

18 i) Write the equation of the line that is parallel to the vector 2**i** + **j** - 3**k** and passes through the point (2, 1, 2). Call this line l_1 .
ii) Write down the coordinates of another point on the line (any other point.)
iii) Does the point with coordinates (3, 1, 6) lie on the line?
iv) Write the equation of the line that passes through the points (3, 1, 6) and (1, 0 , 0). Call this l_2 .
v) If the lines l_1 and l_2 intersect then their equations must be equal. Set the two equations equal and try to solve.
vi) Are lines l_1 and l_2 parallel? If they are not parallel does that mean they intersect?
vii) By making two lines in the air with your fingers show that lines can be neither parallel nor intersecting. (Lines that are not parallel and do not intersect are called "skew".)
viii) Calculate the angle between lines l_1 and l_2. (Hint: the angle between the lines is the same as the angle between the *direction vectors* of each line.)

19) By analogy with the Cartesian equation of the line in 2D ($ax+by=c$), you may have been expecting the equation of a line in 3D to be of the form $ax+by+cz=d$. I know I did when I first looked into this,but it is not the case.
i) Draw x-y-z axes and mark the points X(1, 0, 0), Y(0, 1, 0) and Z(0, 0, 1).
ii) Show that all three of these points fit the equation $x+y+z=1$, but clearly they do not lie on a line.
iii) The three points do not lie on a *line* but they do lie on a *plane*. (Three points in 3D always lie on a plane.) What is the equation of the plane? (Hint: Easy question.)
iv) Pick two or three points that you think lie on the line and substitute them into the equation to see if they do.
v) Pick two or three points that you think *do not* lie on the line and substitute them into the equation to see if they do not.

The Cartesian equation of the plane in 3D is $ax+by+cz=d$. A graphing app such as Geogebra does good 3D graphics allowing you to quickly display lines, planes and polygons. I suggest you enter the equation of a plane into the 3D Geogebra graphing axes and explore them a bit. We are now going to see if we can find the equation of a *plane* expressed in terms of *vectors*.

20 i) Draw a line (any line) in 3D space on x-y-z axes. Draw a plane that is perpendicular to the given line. Draw a set of planes all of which are perpendicular to the given line. (You should now have drawn something like a "kebab" with thin pieces of food stuck on a stick.)
ii) What do all these planes have in common?
iii) If you knew a point on one of the planes would that allow you to uniquely identify the plane?
iv) Mark two points on one of your planes and join them with a vector. What can you say about the vector and the line?
v) Call the points you have marked on the plane P and Q. Mark an origin O and draw position vectors $\mathbf{p}=\overrightarrow{\mathbf{OP}}$ and $\mathbf{q}=\overrightarrow{\mathbf{OQ}}$. The vector from P to Q is **q - p** . Your line will have a vector equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$.
What can you see about vectors **q - p** and **b** ?
vi) What can you say about the scalar product **(q - p).b ?**

21 i) Draw a line and a plane perpendicular to the line. Mark the point of intersection of the line and the plane and call it **a**. Mark a *general* point on the plane with position vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.
Let the line have equation $\mathbf{s}=\mathbf{a}+\lambda\mathbf{b}$, where **s** is a general point on the line. Show that $(\mathbf{r}-\mathbf{a}).\mathbf{b}=0$.
ii) Use your diagram to convince yourself of the truth of these statements:
If P is a general point on the plane with position vector **r ,** then $(\mathbf{r}-\mathbf{a}).\mathbf{b}=0$.
If P is a general point with position vector **r** that is not on the plane **,** then $(\mathbf{r}-\mathbf{a}).\mathbf{b}\neq 0$.

22) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$. Expand out the expression $(\mathbf{r} - \mathbf{a}).\mathbf{b} = 0$ in terms of x, y, and z and show that the vector equation of the plane is equivalent to the Cartesian equation of the plane $ax + by + cz = d$.

In question 21 we showed that the expression $(\mathbf{r} - \mathbf{a}).\mathbf{b} = 0$ was the equation of a plane that was perpendicular to a line meeting the plane at point **a**. We could substitute **b** with any other vector that was normal (i.e. perpendicular) to the plane, and that would be any other vector parallel to **b**. Let's call this vector **n** (because it is *normal* to the plane.) Then $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ is the equation of the plane where **a** is the position vector of any point *on the plane* and **n** is any vector *normal to the plane.*

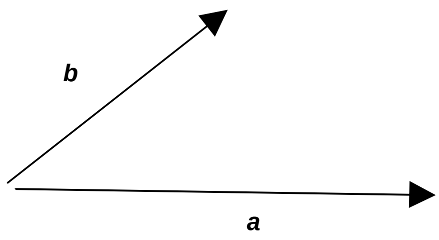
23) Show that the equation of the plane given above is equivalent to the expression **r.n = a.n**, and since **a.n** is a *scalar* we can call it "d" and write **r.n =** d.

In question 23 above, we have introduced a very simple equation of the line: **r.n =** d. **n** is a vector normal to the plane (sometimes "orthogonal" is used instead of "normal". They both mean "perpendicular to.") d = **a.n** where **a** is the position vector of *any* point on the plane. So let's choose the particular point on the plane so that **a** is the *normal vector* from the origin to the plane. In other words **a** is parallel to **n**. And let's make **n** a unit normal vector by dividing **n** by it's magnitude: $\hat{\mathbf{n}} = \dfrac{\mathbf{n}}{|\mathbf{n}|}$. The little "hat" over a vector identifies it as a unit vector (i.e. of length one). So now we have $\mathbf{r}.\hat{\mathbf{n}} = \mathbf{a}.\hat{\mathbf{n}}$ where **a** and $\hat{\mathbf{n}}$ are both normal to the plane.

24 i) If $\mathbf{r}.\hat{\mathbf{n}} = \mathbf{a}.\hat{\mathbf{n}}$ show that $\mathbf{a}.\hat{\mathbf{n}} = |\mathbf{a}|$, and that **|a|** is the *perpendicular distance* of the plane from the origin. (Remember, **a** and $\hat{\mathbf{n}}$ are parallel.)
ii) Show that the equation of a plane can be written as $\mathbf{r}.\hat{\mathbf{n}}$ = d, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\hat{\mathbf{n}}$ is a unit vector normal to the plane and d is the perpendicular distance of the plane from the origin.

25) Consider the plane through the points X(1, 0, 0), Y(0, 1, 0) and Z(0, 0, 1) that we examined in question 19.
i) Show that the vector $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and vector $\mathbf{w} = \mathbf{j} - \mathbf{k}$ are both vectors that lie in the plane.
ii) Show that the vector $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to both vectors **v** and **w** and so is perpendicular to the plane.
iii) Show that if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a general point in 3D space such that (**r - i**).**u** = 0 then **r** will be the position vector of a point on the plane.
iv) Show that the vector equation (**r** - **i**).**u** = 0 is equivalent to the equation **r**.**u** = 1 and this, in turn, is equivalent to the Cartesian equation $x + y + z = 1$.
v) Show that the distance of the plane from the origin is $\dfrac{1}{\sqrt{3}}$.

In question 24 vector **u** is a vector that is perpendicular to both vectors **v** and **w**. We are now going to show you a method for finding a vector that is perpendicular to two other vectors. It is another way of multiplying vectors. The dot product is a useful and consistent form of multiplication for vectors, but it gives us a *scalar*. We want a form of multiplication that gives us a *vector*. We also want this product to represent *area*, which is the natural geometrical interpretation of multiplication. So we want an operation, called $\times$, such that $\mathbf{a} \times \mathbf{b}$ has the form $= |\mathbf{a}||\mathbf{b}|(x)\hat{\mathbf{n}}$, where (x) is a quantity that will allow us to calculate an area and $\hat{\mathbf{n}}$ is a unit vector.



26 i) Vectors **a** and **b** are shown left. Mark the angle between the vectors as θ. Draw a third vector joining the ends of **a** and **b** to make a triangle.

ii) What is the area of this triangle, in terms of |**a**| and |**b**| and θ ? Does that suggest a value for (x) in the formula for $\mathbf{a}\times\mathbf{b}$?
iii) Place your right hand against vector **a**, fingers pointing in the same direction as **a** . Curl your fingers towards **b** . Your thumb should point in the direction of unit vector **n** . Draw vector **n** on vectors **a** and **b**, indicating direction.

The vector product of **a** and **b** is $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n}$, where θ is the angle between **a** and **b** and **n** is a *unit vector orthogonal* to **a** and **b**.
Note that since **n** is orthogonal to the other two vectors this implies three dimensions. The *dot product* of vectors is valid in any number of dimensions. *The cross product of vectors is valid only in three dimensions.*

27 i) Consider $\mathbf{b}\times\mathbf{a} = |\mathbf{b}||\mathbf{a}|\sin(\theta)\mathbf{m}$, where **m** is a *unit vector orthogonal* to **a** and **b**. Place your right hand against vector **b** **,** fingers pointing in the same direction as **b** . Curl your fingers towards **a** . Your thumb point should point in the direction of unit vector **m** .
ii) Show that vector **m** = **-n** .
iii) Show that $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$.

28 i) **a** is a vector of modulus 3 and **b** is a vector of modulus 2. The angle between **a** and **b** is 30 degrees. Find $|\mathbf{a}\times\mathbf{b}|$.
ii) Find the area of the triangle defined by **a** and **b** in the diagram.
iii) Copy the diagram above and repeat vectors **a** and **b** to make a parallelogram. What is the area of the parallelogram?

29) **i, j, k** are our base unit vectors.
i) Show that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$.
ii) Show that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$ and give the values of $\mathbf{i}\times\mathbf{k}, \mathbf{j}\times\mathbf{k}, \mathbf{j}\times\mathbf{i}, \mathbf{k}\times\mathbf{j}, \mathbf{k}\times\mathbf{i}$. (Be careful here, you must apply that right hand rule correctly.)
iii) Complete the table below and draw a similar table for the dot product. Note that the order is row x column. So the entry in the table shows us that $\mathbf{i}\times\mathbf{j} = \mathbf{k}$. What is $\mathbf{j}\times\mathbf{i}$?

$\times$	*i*	*j*	*k*
i	*0*	*k*	
j			
k			

30) Any vectors **a** and **b** can be defined in terms of the unit vectors $\mathbf{a} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $\mathbf{b} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$. In that case $\mathbf{a}\times\mathbf{b} = (x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})\times(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})$. Assume that all the normal rules of arithmetic apply here and show that $\mathbf{a}\times\mathbf{b} = (x_2y_3 - x_3y_2)\mathbf{i} - (x_1y_3 - x_3y_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$.

31) Do those values for $\mathbf{a}\times\mathbf{b}$ in question 28 look familiar to you?
Recall from the previous chapter the formula for the determinant of a 3×3 matrix.

Calculate $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$. Note that **i, j, k** in the top row are *vectors*. (The components of a matrix can be vectors as well as scalars.) Compare your answer with the formula in question 28. Conclude that $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$. (This is another reason why the cross product works only in three dimensions.)

32) Three points in 3D space are P(1,1,1) , Q(-1,2,3), and R(0,-1,-1). Find the area of the triangle PQR. (Hint: What is the vector that goes from P to Q? What is the vector that goes from P to R?)

There are some natural questions you will want to ask about planes and lines in space:

- Do two lines intersect and if so where?
- Do a line and a plane intersect and if so at what point and what angle?
- Do two planes intersect and if so at what line and what angle?
- What is the shortest distance between a point and a line or a point and a plane?
- What is the shortest distance between two lines, two planes or a line and a plane?

Before we look at methods for these questions we will show another method for displaying a vector. At the start of this chapter we said that "a vector is a one-column matrix. It can also be considered as an arrow. It can also be considered as an extended form of a "number" in its own space with its own form of arithmetic." We have seen vectors as arrows and vectors in terms of base vectors **i**, **j** and **k**.

The vector a**i** + b **j** +c**k** can also be displayed as the column matrix $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

The dot product of vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is written $\begin{pmatrix} a \\ b \\ c \end{pmatrix}.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax+by+cz$.

33) Let L_1 be the line with equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and L_2 the line with equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.

i) These lines are not parallel. (How do you know?) That does not necessarily mean they intersect. Demonstrate with two fingers how two lines can be neither parallel nor intersecting. This state is called "*skew*".

ii) If the lines intersect then the equation $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ has a solution. Find the values of λ and μ and the point of intersection of the lines.

iii) Show that the lines with equations $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ are neither parallel nor do they intersect. Therefore they are skew.

iv) Find the angle between the two lines.

34) The plane we will call plane π has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \left(\frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right) = 4$.

i) Does the line with cartesian equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix}$ intersect plane π and if so, where?

ii) Find the acute angle between the line and the plane. (Hint: find the angle between the direction vector of the line and the normal vector of the plane . Because the normal vector is *perpendicular* to the plane you will want the *complementary* angle.)

iii) Show that the line with equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ does *not* intersect the plane π .

35 i) Find the perpendicular distance between the plane π and the origin. (Hint: Easy question.)
ii) **a = i** - 3**j** + 4**k**, **n = i - j +** 3**k** and (**r - a**).**n** = 0 is the equation of plane ρ .
Show that the plane ρ is parallel to plane π . Find the perpendicular distance between the plane ρ and the origin. Hence find the perpendicular distance between planes π and ρ .

36 i) Show that the point **b =** 2**i - j -** 2**k** does not lie on plane ρ .
ii) Find the shortest distance from the point to the plane. (Hint: find the equation of the plane through **b** that is parallel to plane ρ . Then find the distance between the plane through **b** and the plane ρ . Note that **b.n** will be *negative*. What does that tell you about the position of the two planes?)

37 i) Show that the point **b** in question 34 does not lie on the line with equation $\mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{k})$.
ii) Find the shortest distance from the point to the line. (Hint: Find the vector from **b** to **r** . Find the length of this vector as a function of λ . Find the minimum value of this function.)

38) In question 33 you showed that the the lines with equations $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ were skew.
i) Using your fingers as two skew lines, convince yourself that there is always a line between two skew lines that is perpendicular to both lines.
ii) Show that a unit vector normal to both lines will be $\frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}$.

iii) If P is a general point on a line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and Q is a general point on a line with equation $\mathbf{s} = \mathbf{c} + \mu\mathbf{d}$, show that $\overrightarrow{PQ} = (\mathbf{c} - \mathbf{a}) + (\mu\mathbf{d} + \lambda\mathbf{b})$ and this is a vector joining the two lines..

v) At the beginning of this chapter we showed that the projection of of vector **a** on vector **b** may be written as $(\mathbf{a}.\hat{\mathbf{b}})\hat{\mathbf{b}}$. Hence show that the projection of $\overrightarrow{PQ}$ onto the normal vector $\hat{\mathbf{n}} = \frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}$ is $\left(\left((\mathbf{c} - \mathbf{a}) + (\mu\mathbf{d} + \lambda\mathbf{b})\right).\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$ (because $\mathbf{b}.(\mathbf{b} \times \mathbf{d}) = \mathbf{d}.(\mathbf{b} \times \mathbf{d}) = 0$) and that this simplifies to $(\mathbf{c} - \mathbf{a}).\left(\frac{\mathbf{b} \times \mathbf{d}}{|\mathbf{b} \times \mathbf{d}|}\right)$ and therefore the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{s} = \mathbf{c} + \mu\mathbf{d}$ is $\frac{|(\mathbf{c} - \mathbf{a}).(\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$.

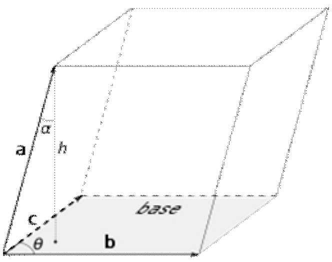
vi) Hence find the shortest distance between lines with equations $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\mathbf{s}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$.

39 i) If **a, b, c** are the position vectors of points P(1,1,1), Q(-1,2,3), and R(0,-1,-1) respectively, calculate $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}.(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}.(\mathbf{a}\times\mathbf{b})$. (Hint: remember that (eg) $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$) What do you notice?
ii) Show that for general vectors unit vectors, $\mathbf{a}=x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$, $\mathbf{b}=y_1\mathbf{i}+y_2\mathbf{j}+y_3\mathbf{k}$ and $\mathbf{c}=z_1\mathbf{i}+z_2\mathbf{j}+z_3\mathbf{k}$, $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\mathbf{b}.(\mathbf{c}\times\mathbf{a})=\mathbf{c}.(\mathbf{a}\times\mathbf{b})$.

iii) Show that $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3\end{vmatrix}$.

iv) Show that $\mathbf{a}.(\mathbf{a}\times\mathbf{b})=0$ and $\mathbf{b}.(\mathbf{a}\times\mathbf{b})=0$.

$\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ is called the *scalar triple product.*
A *parallelepiped* is the 3D extension of a parallelogram. In other words it is a "slanted box" whose sides are parallelograms. Copy the diagram of a parallelepiped given below.



(copyright Wikipedia, Parallelepiped.)

The volume of the parallelepiped is "(area of base) times height". The area of the base is equal to $|\mathbf{b}\times\mathbf{c}|$. The altitude h is perpendicular to the plane of **b** and **c**. So h is parallel to the vector **n**, where $\mathbf{b}\times\mathbf{c}=|\mathbf{b}||\mathbf{c}|\sin(\theta)\mathbf{n}$. α is the angle between vector **a** and altitude h. So therefore $h=\mathbf{a}\cos(\alpha)$ and therefore the volume of the parallelepiped is $|\mathbf{b}\times\mathbf{c}||\mathbf{a}|\cos(\alpha)$. But this is exactly the definition of $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$. So the volume of the parallelepiped is $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$.

40) With vectors **a, b** and **c** the same as in question 39, calculate the volume of the parallelepiped.

41) A *pyramid* is a 3d shape formed by a base with edges that meet at a single point at the top. A *quadrangular pyramid* is a pyramid whose base is a four sided figure.
i) Draw a quadrangular pyramid.
ii) The volume of a quadrangular pyramid is "⅓ (area of base) times height". Use a similar argument as for the parallelepiped to show that the volume of the quadrangular pyramid is $\frac{1}{3}|\mathbf{a}.(\mathbf{b}\times\mathbf{c})|$.

42) A *tetrahedron* is a solid of four faces where each face is a triangle. So a *tetrahedron* is a *triangular pyramid*.
i) Draw a tetrahedron.
The volume of a tetrahedron is "⅓ (area of base) times height". Use a similar argument as for the parallelepiped to show that the volume of the tetrahedron is $\frac{1}{6}|\mathbf{a}.(\mathbf{b}\times\mathbf{c})|$.

43) A *regular* tetrahedron is one in which all four faces are equilateral triangles. What is the volume of a regular tetrahedron with sides of length 1?
(Hint: The height of the tetrahedron is $\frac{1}{3}\sqrt{6}$ and the altitude is placed at the intersection of the three medians of the base. Draw the tetrahedron as viewed from above with one corner at (0,0,0). Show that three points of the tetrahedron are at $\mathbf{i}$, $\frac{1}{2}\mathbf{i}+\frac{1}{2}\sqrt{3}\mathbf{j}$ and $\frac{1}{2}\mathbf{i}+\frac{1}{6}\sqrt{3}\mathbf{j}+\frac{1}{3}\sqrt{6}\mathbf{k}$. Then apply the formula.)