Chapter 12: Matrices III - Eigenvalues and Eigenvectors

1) Imagine a plane in 3D space created by rotating the plane of the x-y axis 45 degrees upwards towards the z axis, with the x axis as the axis of rotation.
i) Draw a diagram of the plane in the x-y-z axes. Mark three points on the plane and the triangle of vectors formed by the three points.
ii) Call this plane π and write its equation, in vector form and in cartesian form.
iii) Mark the unit vectors **i, j, k** on your diagram and show the image of each vector under reflection in the plane π. One unit vector is unchanged. Which one? What is the image of **j** and the image of **k?**
iv) Write the matrix whose columns are composed of the image of vectors **i, j** and **k** after reflection in plane π. This is your Transformation matrix. Call it T.
v) What is the image of point P(1,0,1) under the transformation?
vi) Consider the line with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$. Write the image of the line under transformation T.
vii) Every transformation has an *inverse transformation* which reverses it. It is no surprise that the inverse of transformation matrix T is the inverse transformation matrix T^{-1}. Show that, in this case, T^{-1} is T, so that $\mathrm{T}^2 = \mathrm{I}$. Why would you expect this to be the case?
viii) The plane π has the simple vector equation $\mathbf{r} = \lambda\mathbf{i} + \mu(\mathbf{j} + \mathbf{k})$. Since this is the plane of reflection we would expect that this plane would remain the same under transformation T. Check that this is the case.

You can create all the objects above and perform the relevant transformations with a suitable graphing app such as Geogebra. It is useful and satisfying to see the results on screen. I suggest you try this and spend a bit of time exploring different shapes and different transformations with your favourite graphing app.

In question 1 we showed that any point on the plane π would be unaffected by transformation T, because π was the plane of reflection (i.e. the mirror.) This means that if **r** was the position vector of a point on the plane we would expect that T**r = r**. We say that **r** is *invariant* under the transformation. Invariant points are points of interest in any transformation. More widely, we are interested in points **r** such that $\mathrm{T}\mathbf{r} = \lambda\mathbf{r}$ for some constant λ. Then shape is preserved though scale may change.

2) We want to find vectors **r** such that $\mathrm{T}\mathbf{r} = \lambda\mathbf{r}$, where T is the reflection in question 1.
i) Show that $\mathrm{T}\mathbf{r} = \lambda\mathbf{r}$ means $(\mathrm{T} - \lambda\mathrm{I})\mathbf{r} = 0$.
ii) Show that $(\mathrm{T} - \lambda\mathrm{I})\mathbf{r} = 0$ implies $|\mathrm{T} - \lambda\mathrm{I}| = 0$. (Hint: If $|\mathrm{T} - \lambda\mathrm{I}| \neq 0$ that means $(\mathrm{T} - \lambda\mathrm{I})^{-1}$ exists. So what would $(\mathrm{T} - \lambda\mathrm{I})^{-1}(\mathrm{T} - \lambda\mathrm{I})\mathbf{r}$ equal?)

3 i) With T the matrix in question 1, show that $|\mathrm{T} - \lambda\mathrm{I}| = 0$ implies that $(\lambda - 1)^2(\lambda + 1) = 0$, and so λ =1 or -1.
ii) Show that when λ =1 , the equation $(\mathrm{T} - \lambda\mathrm{I})\mathbf{r} = 0$ implies that $\mathbf{r} = \alpha\mathbf{i} + \beta\mathbf{j} + \beta\mathbf{k}$, where α and β are any two numbers, not both zero.
When λ = -1 , the equation $(\mathrm{T} - \lambda\mathrm{I})\mathbf{r} = 0$ implies that $\mathbf{r} = \beta\mathbf{j} - \beta\mathbf{k}$, where β is any non-zero number.
iii) Mark these vectors as an arrow on your sketch of the plane.
(It's not always easy to sketch a 3D diagram on paper but do the best you can. And also use a graphing app, like Geogebra, to picture it for you.)

These vectors are called the "*eigenvectors*" of the matrix T and the values of λ are called the "*eigenvalues*". Together the eigenvector (**r**) and eigenvalue (λ) give a solution to the equation $\mathrm{T}\mathbf{r} = \lambda\mathbf{r}$.

4) Label the eigenvectors $\mathbf{e}_1 = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \beta\,\mathbf{k}$ and $\mathbf{e}_2 = \chi\,\mathbf{j} - \chi\,\mathbf{k}$. Label the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

i) State the geometric relationship between eigenvector $\mathbf{e}_1$ and the plane and between eigenvector $\mathbf{e}_2$ and the plane.
ii) Could you have predicted that these would be the vectors before you calculated them?

5) Note that $\mathbf{e}_1$ has two independent variables α and β. This means it can be split into two independent eigenvectors, $\mathbf{e}_0 = \alpha\,\mathbf{i}$ and $\mathbf{e}_1 = \beta(\mathbf{j}+\mathbf{k})$ and $\mathbf{e}_2 = \chi(\mathbf{j}-\mathbf{k})$.
i) Show that $\mathbf{e}_0.\mathbf{e}_1 = 0, \mathbf{e}_0.\mathbf{e}_2 = 0$ and $\mathbf{e}_1.\mathbf{e}_2 = 0$. This means that the three eigenvectors are perpendicular to each other.
We call perpendicular eigenvectors "*orthogonal*".
We will see that eigenvectors are more useful when they have length one. So they are unit vectors. We call unit eigenvectors "*normalised*".
ii) "Normalise" the three eigenvectors by dividing them by their modulus.
iii) Show that $\mathbf{e}_0.\mathbf{e}_0 = \mathbf{e}_1.\mathbf{e}_1 = \mathbf{e}_2.\mathbf{e}_2 = 1$ and $\mathbf{e}_0.\mathbf{e}_1 = \mathbf{e}_0.\mathbf{e}_2 = \mathbf{e}_1.\mathbf{e}_2 = 0$ and that these equations may be summarised by the single equation $\mathbf{e_i}.\mathbf{e_j} = \{1 \text{ if } \mathbf{i} = \mathbf{j}, 0 \text{ if } \mathbf{i} \neq \mathbf{j},\ \mathbf{i}, \mathbf{j} = 1, 2, 3\}$.
This single equation shows the eigenvectors are both *orthogonal* and *normalised*. So we call them "*orthonormal*" eigenvectors.

6 i) Write eigenvectors $\mathbf{e}_0, \mathbf{e}_1$ and $\mathbf{e}_2$ as *column* vectors.
ii) Create the eigenvector matrix, E, whose columns are each of the eigenvectors $\mathbf{e}_0, \mathbf{e}_1$ and $\mathbf{e}_2$.
So $\mathrm{E} = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}$.
Matrix E has columns which are orthogonal and normalised so we call E an "orthonormal" matrix.

iii) Show that the transpose of E is $\mathrm{E}^{\mathrm{T}} = \begin{pmatrix} \mathbf{e}_0{}^{\mathrm{T}} \\ \mathbf{e}_1{}^{\mathrm{T}} \\ \mathbf{e}_2{}^{\mathrm{T}} \end{pmatrix}$.

iv) If E = E^{T} then E is a "symmetric" matrix. If $\mathrm{E}^2 = \mathrm{I}$ then E is a "self-inverse" matrix.
Show that E is a *symmetric* and *self-inverse* matrix.

7) Matrix E is the matrix of eigenvectors of matrix T - the reflecting transformation matrix that we started with.
i) Create the product of matrices $\mathrm{E}^{\mathrm{T}}\mathrm{TE}$. Show that this matrix is zero in all elements except those on the diagonal and the elements on the diagonal are the eigenvalues of T.
Cool! How did that happen?
ii) Show that, in general, if matrix A has eigenvectors $\mathbf{e}_0, \mathbf{e}_1$ and $\mathbf{e}_2$ and these are *orthonormal* eigenvectors, and matrix P is the matrix whose columns are $\mathbf{e}_0, \mathbf{e}_1$ and $\mathbf{e}_2$ then $\mathrm{PAP}^{\mathrm{T}}$ will be a matrix with zero in all elements except those on the diagonal and the elements on the diagonal are the eigenvalues of A.

Not surprisingly, a matrix with zeroes everywhere except on the diagonal is called a *diagonal matrix*. "Diagonalising" a matrix is a key method for certain processes such as solving certain kinds of differential equations and in "Operator Theory" which is the key mathematics of quantum mechanics.

The statement in 7 (ii) above is a statement of how to diagonalise a matrix. To diagonalise matrix A - find the eigenvectors, make matrix P from the eigenvectors and $\mathrm{PAP}^{\mathrm{T}}$ is the diagonal matrix.
Note that this can only be done when A is a *symmetric* matrix. In this case E is also a symmetric matrix, but the eigenvector matrix E is not symmetric in general.

10) Let's look at another transformation. U transforms the general point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ into $\begin{pmatrix} x+y \\ 2x+y \\ 3(x+y+z) \end{pmatrix}$.

i) Write down the transformation matrix U.

ii) Show that the three eigenvalues are $3, 1+\sqrt{2}, 1-\sqrt{2}$ and the matching eigenvectors are

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{e}_2 = \alpha \begin{pmatrix} 1 \\ \sqrt{2} \\ \frac{1}{\alpha} \end{pmatrix} \text{ and } \mathbf{e}_3 = \alpha \begin{pmatrix} -1 \\ -\sqrt{2} \\ \frac{1}{\alpha} \end{pmatrix}$$ and state the value of α.

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iii) Show that these eigenvectors are *not orthogonal.*

12 i) Find the three independent eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$.

(Hint: One eigenvalue is $\lambda = 3$).

ii) Show that the eigenvectors are orthogonal and normalise them.

iii) Diagonalise the matrix.