Chapter 13: Polar Coordinates

1) Draw these loci on the complex plane:
i) z, where Mod(z) = 2.
ii) z, where Arg(z) = $\frac{\pi}{4}$.
iii) z, where Mod(z) = Arg(z).
(Hint: Plot values of z where both Mod(z) and Arg(z) are equal to $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi \cdots$
Keep on going and draw a smooth curve through the points.)

The loci you drew in question 1 should have been, respectively, a circle centre the origin, a line extending from the origin at 45 degrees above the horizontal and the last is a nice spiral expanding out from the origin. Since z can be written as $re^{i\theta}$ where r = Mod(z) and θ = Arg(z) each locus can be written as a curve in terms of r and θ, where r is the distance from the origin and θ is the angle above the horizontal. So we can consider these curves as being drawn in the usual Cartesian plane but in tems of r and θ, not x and y.

2) Write the equation of each of the curves in question 1 in the form $r = f(\theta)$.
Congratulations. You have just drawn three curves in "*polar coordinates*".

3 i) Mark point P(x, y) on x-y axes. Draw a line from O to P and call the length of the line "r". Mark the angle the line makes with the x-axis as θ.
ii) Show that when a point is defined in terms of r and θ, not x and y, then

$$r^2 \equiv x^2 + y^2,\ \cos(\theta) \equiv \frac{x}{r},\ \sin(\theta) \equiv \frac{y}{r},\ \tan(\theta) \equiv \frac{y}{x}$$

4) Install in your phone a graphing app. "Desmos" or "Geogebra" are examples.
Enter the following curves for sketching:
i) $r = a$, where a has a range of values,
ii) $\theta = a$, where a has a range of values,
iii) $r = a\theta$, where a has a range of values,
Remark on what effect the parameter "a" has on the curve

5) Enter the following curves in your graphing package:
i) $r = \cos(a\theta)$, where a has a range of values .
ii) $r = a\cos(b\theta)$, where a and b have a range of values .
iii) $r = a + b\cos(\theta)$, where a and b have a range of values .
Note that you can make a "slider" for parameters a and b. I recommend that you do.

6) In each case of question 4 and 5, copy the curve into your notes. Note the *x* and *y* intercepts of each curve. Remark on what effect parameters *a* and *b* have on the shape of the curve.

6) Enter the curves in question 3 with "sin" in place of "cos". What difference does this make? Is it consistent across curves i to iii?

7) Enter the curves with tan, cot, sec and cosec in place of cos. What difference does this make? Is it consistent across curves i to iii?

Note: r must always be greater than zero. So what happens with, for example, $r = \cos(\theta)$ when $\theta > \frac{\pi}{2}$? (The convention is: when $r < 0$ replace θ with $\theta + \pi$.)

8) The curves $r = \sin(\theta), r = \cos(\theta), r = \cos ec(\theta), r = \sec(\theta)$ have a particularly simple form. What are they? Explain why, by converting the polar equation to a cartesian equation.

9) Write the Polar Form of the parabolic curve $y = x^2$. Check your answer by putting the equation into your graphing package.

10) Enter the graph $r = e^{a\theta}$ into your graphing package, with a range of values of *a*. (You can use a "slider" for *a*.) Compare it with the other spiral graph $r = a\theta$.

R E 11) The graph $r = e^{a\theta}$ is called the "logarithmic spiral" or the "golden spiral". It has a long history and many interesting properties that are worth investigating.

Although our curves are defined in terms of r and θ, they are still displayed on the *x*-*y* coordinate plane. So the gradient of the curve is given by $\frac{dy}{dx}$, although $\frac{dy}{dx}$ itself will usually be expressed in terms of r and θ.

12 i) Noting that $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and that $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, show that

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta) .$$

ii) Write the equivalent expression for $\frac{dx}{d\theta}$.

iii) Hence show that $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$.

13 i) Using your graphing software, plot the curves
$r = 2 + \cos(\theta),\ r = 1 + \cos(\theta),\ r = 1 + 3\cos(\theta),\ 0 < \theta < 2\pi$.

ii) You should see three distinct curves - an oval shape or "ellipse", a heart shape, called a "cardioid" and a "dimpled ellipse", where the "dimple" is a dent on the left hand side of the ellipse. Copy these shapes and their equations into your notes.

14) Consider the curve with the general equation $r = p + q\cos(\theta)$.

i) Show that if $p \le q$ then r can equal zero and we get the cardioid. (For what values of θ does r equal zero?)

ii) Show that if $p > q$ then r cannot be zero and we will get an ellipse, with or without a dimple.

15) On each of your curves in question 13 note the points where the tangents to the curves are vertical. In the oval shape there are two points where tangents are vertical - one at each end. In the cardioid there are three points and in the dimpled ellipse there are four. Draw the vertical tangents in each case.

16) A tangent will be vertical when $\frac{dx}{d\theta} = 0$. Why? Find the points where $\frac{dx}{d\theta} = 0$ for each of the three curves. Show that these points match the vertical tangents you have drawn. Show that there will be a dimple on the ellipse if $q < p \le 2q$.

17) A tangent will be horizontal when $\frac{dy}{d\theta} = 0$. Why? Find the points where $\frac{dy}{d\theta} = 0$ for each of the three curves. Show that these points match horizontal tangents on your curve and draw these tangents.

In Polar Coordinates the x-axis is known as the "initial line". (Why?) So a horizontal tangent is usually described as being "parallel to the initial line". A vertical tangent is usually described as being "perpendicular to the initial line".

18 i) On a graphing app, plot the Logarithmic Spiral with equation $r = e^{a\theta}$
for a range of values of a .

ii) Show that vertical tangents are at $\tan(\theta) = a$ and horizontal tangents are at $\tan(\theta) = -\frac{1}{a}$

iii) Show that the graph of $r = e^{\theta}$ crosses the horizontal and vertical axes at 45° to the axis.

19 i) On a graphing app, plot the Logarithmic Spiral with equation $r = a\,e^{\theta}$
for a range of values of a .

ii) Show that vertical tangents are at $\tan(\theta) = \frac{1}{\theta}$ and horizontal tangents are at $\tan(\theta) = -\theta$.

iii) By sketching graphs of $y = \tan(\theta)$ and $y = \theta$, or otherwise, show that the spiral inreasingly approaches horizontal tangents on the vertical axis and vertical tangents on the horizontal axis as it spirals out from the origin.

You have found out how to find derivatives of curves in polar coordinates. If we want to find the area within curves in polar coordinates we need to find integrals.

20 i) Sketch the curve with equation $r = \theta$, where $0 \le \theta \le \frac{\pi}{2}$ and shade the area enclosed by the curve and the x and y axes.

ii) Show that this area can be considered as the sum of many small sectors of radius r and angle $d\theta$.

Therefore the area of the small sector is $\frac{1}{2}r^2 d\theta$ and the total area will be $\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2 d\theta$.

In this case $r = \theta$, so substitute for r in the integral and calculate the area.

iii) Show that the area enclosed by the curve and the x and y axes when the curve is $r = \theta$ is $\frac{1}{48}\pi^3$.

iv) Show that the area enclosed by the curve and the x and y axes when the curve is $r = e^{\theta}$ is $\frac{1}{4}(e^{\pi} - 1)$.

21 i) Sketch the curve with equation $r = \cos(\theta)$ and the curve with equation $r = \sin(\theta)$.

Shade the area where the two curves overlap.

ii) Show that the curves intersect at the points where and $\theta = 0$ and $\theta = \frac{\pi}{4}$. Hence show that the total area is $\int_0^{\frac{\pi}{2}} \sin^2(\theta) d\theta$ or $\int_{\frac{\pi}{2}}^{0} \cos^2(\theta) d\theta$, whichever you like.

iii) Calculate both integrals, showing that they give the same value.