Chapter 14: Hyperbolic Functions

Let us define two new functions which, for the moment, we will call $c(x)$ and $s(x)$.
$c(x)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$ and $s(x)=\frac{1}{2}\left(e^{x}-e^{-x}\right)$.

1) Show that $c(x)$ is an even function and $s(x)$ is an odd function. (Remember, a function $f(x)$ is *even* if $f(-x)=f(x)$ and *odd* if $f(-x)=-f(x)$. A function is not necessarily even or odd.)

2) Sketch on separate curves the functions $y=c(x)$ and $y=s(x)$. (Remember, the curve of an *even* function is symmetric by reflection in the y-axis. The curve of an *odd* function is symmetric by a 180° rotation about the origin.)

3) Show that $\frac{d}{dx}c(x)=s(x)$ and $\frac{d}{dx}s(x)=c(x)$.

4) Write down the Taylor Series for e^{x} and for e^{-x} and use these to write the Taylor Series for $c(x)$ and $s(x)$. Do these series remind you of other ones?

5) Show that the Taylor Series for $c(ix)$ is the same as that for $\cos(x)$ and the Taylor Series for $s(ix)$ is the same as that for $i\times\sin(x)$. Confirm by substituting in to the definition of the function that $c(ix)=\cos(x)$ and $s(ix)=i\times\sin(x)$. Conclude that any complex number z may be written as $c(ix)+s(ix)$.

6) i) Show that $\left(e^{x}\pm e^{-x}\right)\left(e^{y}\pm e^{-y}\right)\equiv\left(e^{(x-y)}+e^{-(x-y)}\right)\pm\left(e^{(x+y)}+e^{-(x+y)}\right)$

and that $\left(e^{x}\pm e^{-x}\right)\left(e^{y}\mp e^{-y}\right)\equiv\left(e^{(x-y)}-e^{-(x-y)}\right)\mp\left(e^{(x+y)}-e^{-(x+y)}\right)$.

ii) Hence, show that $c(x)c(y)+s(x)s(y)\equiv c(x+y)$ and $s(x)c(y)+c(x)s(y)\equiv s(x+y)$.

iii) By taking $x=y$ in the identities above show that $c(2x)\equiv c^{2}(x)+s^{2}(x)$, $s(2x)\equiv 2s(x)c(x)$ and $c^{2}(x)-s^{2}(x)\equiv 1$.

Do these identities remind you of other, similar ones?

Of course by now you will have realised that the functions $c(x)$ and $s(x)$ have properties almost the same as those of $\cos(x)$ and $\sin(x)$, even though the curves look very different. And we did not invent $c(x)$ and $s(x)$. They date from the 1760's when they were introduced independently by the Italian mathematician Vincenzo Riccati and the French mathematician Johann Heinrich Lambert. Riccati used the names "Sc" and "Cc" to for the functions we know as sine and cosine, because these were known as the "circular" functions. He used the names "Sh" and "Ch" for the functions I have introduced as $s(x)$ and $c(x)$, but $s(x)$ is properly known as $\sinh(x)$ and $c(x)$ is properly known as $\cosh(x)$. The "h" stands for "hyperbolic" and these functions are generally known as the "hyperbolic" functions. You will have to study the FP3 "Extra Pure" module to see why.

The full set of trigonometric (or "circular") functions is sin, cos, tan, cosec, sec and cot. The last three are the "reciprocal trigonometric functions", obtained by taking the reciprocal of each of the first three. The corresponding set of "hyperbolic" functions is sinh, cosh, tanh, cosech, sech and coth. Sinh is pronounced "shine", tanh is pronounced "than", cosech is pronounced "cosesh" and "sech" is pronounced "sesh". "cosh" and "coth" are pronounced as spelled.

The functions tanh, cosech, sech and coth are defined in terms of sinh and cosh in exactly the same way that the trigonometric functions tan, cosec, sec and cot are defined in terms of sin and cos.

7) Since $\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}$ and $\coth(x)=\dfrac{\cosh(x)}{\sinh(x)}$ show that $\tanh(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ and $\coth(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$.

8 i) Find sinh(ln(2)), cosh(ln(2)) and tanh(ln(2)).
ii) Find sech(ln(2)), cosech(ln(2)) and coth(ln 2)).
Your calculator should be able to give you these values. Look for a “hyp” button on your calculator.

9 i) Sketch the curve of each of the functions $\tanh(x)$, $\coth(x)$, $\operatorname{sech}(x)$ and $\operatorname{cosech}(x)$ on paper and check with your graphing app or calculator. (Note that all of these curves have asymptotes. For each curve, state the equation of the asymptote and draw it before fitting the curve.)
ii) What is the domain and range of each function?

We have seen how the trigonometric identities in sin and cos have counterpart identities in sinh and cosh. You will know certain identities involving the circular functions (sin, cos, tan, sec, cosec and cot.) Some of these identities are in the formula book.
10) Write down all the identities you know about the trigonometric functions and prove them.

These identities also have similar counterparts in the hyperbolic functions. Some of these identities are exactly the same as the circular functions (with the addition of “h” to the end of the function name) and some are slightly different.
11) Write down the hyperbolic identity next to its trigonometric counterpart and prove the identity from the definition.

12) Compare your trigonometric identities with your hyperbolic identities.
Test the truth of, and justify, the following statement. *Each trigonometric identity has a corresponding hyperbolic identity obtained by replacing cos by cosh and sin by sinh,* ***except*** *that any product or implied product of two sine terms must be replaced by* ***minus*** *the product of two sinh terms.*
(The above is known is “Osborn’s Rule” and was published in The Mathematical Gazette, Vol 2, No 34, July 1902.)

13) You will know the derivatives of the six trigonometric functions, and some of these are given in your formula book. Create a table like the one below with the derivative of all six trigonometric functions. Prove the derivative where required.

y (= trig. function)	$\dfrac{dy}{dx}$
sin	cos
cos	-sin

Derivatives and integrals of the hyperbolic functions are closely related to derivatives and integrals of the trigonometric functions, and are also given in the formula book.

14 i) Write a table next to the one above but for the hyperbolic functions. Prove the derivative in each case.
ii) Test the truth of, and justify, the following statement. *The derivatives of the hyperbolic functions are the equivalent of plus or minus the derivative of the circular function. The latter case (minus) occurs whenever the derivative of cosh(x) is required.*

15 i) Of course the table works the other way, so that the function on the left is the integral of the function on the right. These integrals should be given in your formula book. Check that they are.
ii) Two integrals not given in the list above are the integrals of $\tanh(x)$ and $\coth(x)$. Show by differentiating that the integral of $\tanh(x)$ is $\ln(\text{sech}(x))$ and the integral of $\coth(x)$ is $-\ln(\text{cosech}(x))$.

All functions have inverses and it will be no surprise to hear that sinh, cosh and tanh have inverses too. They are known as arsinh, arcosh and artanh. You might wonder why they are not arcsinh, arccosh and arctanh like the trig, functions. Well, in some books you will see this. (You might even see the names argsinh, argcosh and argtanh, where "arg" is short for "argument".) We showed in Chapter 10 (Question 9) that the reason for the "arc" in "arcsin" is that the inverse sin function equates to the length of an arc. The "ar" in "arsinh" is short for "area" because the inverse sinh and cosh functions equates to the area under a curve, but you will need to study hyperbolic curves in the FP3 option to see why.

16) Sketch the graph of $y=\sinh(x)$ and reflect in $y = x$ to get the curve of $y=\text{arsinh}(x)$. State the domain and range of $\text{arsinh}(x)$. Check your curves with your graphing app.

17) Sketch the graph of $y=\cosh(x)$, $x\ge 0$ and reflect in $y = x$ to get the curve of $y=\text{arcosh}(x)$. State the domain and range of $\text{arcosh}(x)$.

18) Sketch the graph of $y=\tanh(x)$, $x\ge 0$ and reflect in $y = x$ to get the curve of $y=\text{artanh}(x)$. State the domain and range of $\text{artanh}(x)$.

19) Sketch the curves of $y=\text{sech}(x)$, $y=\text{cosech}(x)$, $y=\coth(x)$ and their inverses. State the domain and range of each function. Check your curves with your graphing app.

If $f(x)$ is a function and $y=f(x)$ then the inverse function is the unique function, written $f^{-1}(x)$, such that $f\left(f^{-1}(x)\right)=f^{-1}\left(f(x)\right)=x$. In other words, $f^{-1}(x)$ "undoes" $f(x)$. We usually find this function $f^{-1}(x)$ by writing $y=f(x)$ and then rearranging the formula until we have $x=g(y)$.
Then $g(x)=f^{-1}(x)$.

20 i) Write $y=\sinh(x)=\tfrac{1}{2}\left(e^{x}-e^{-x}\right)$. Rearrange this to get the quadratic formula $e^{2x}-2ye^{x}-1=0$. Show that this has the solution $e^{x}=y+\sqrt{y^{2}+1}$ and conclude that $\text{arsinh}(x)=\ln\left(x+\sqrt{x^{2}+1}\right)$.

ii) Repeat the method above to show that $\text{arcosh}(x)=\ln\left(x+\sqrt{x^{2}-1}\right)$, $x\ge 1$.

iii) Repeat the method above to show that $\text{artanh}(x)=\tfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $x<1$.

We use two methods to solve a hyperbolic equation. The first way is to break it down into e^{x} and e^{-x} and make it an exponential equation. The second way is the same way we solve a trigonometric equation - by rearranging and applying identities until we end up with $\sinh / \cosh / \tanh(x)=a$ where a is a real number. Then we use the inverse.

21) Look again at the graphs of sinh, cosh and tanh and use them to answer the following questions.
i) How many solutions are there to the equation $\sinh(x)=a$ where a is any real number?
ii) Show that the equation $\cosh(x)=a$ has two solutions when $a>1$, one solution when $a=1$ and no solutions otherwise.
iii) Show that the equation $\tanh(x)=a$ has one solution when $|a|<1$.

22 i) Show that $3\sinh(x)+4\cosh(x)=\frac{7}{2}e^{x}+\frac{1}{2}e^{-x}$.
iii) Hence solve the equation by converting it to a quadratic exponential equation.

23 i) Use hyperbolic identities to show that the equation $3\sinh(x)+4\cosh^2(x)=5$ is equivalent to $4\sinh^2(x)+3\sinh(x)-1=0$.
ii) Hence solve the equation as a quadratic in $\sinh(x)$.

24 i) If, in general, $ax+by=k(x+y)+l(x-y)$, where x and y are variable, show that $ax=(k+l)x$ and $by=(k-l)y$ and so $k=\frac{1}{2}(a+b)$ and $l=\frac{1}{2}(a-b)$.
ii) Hence show that the equation $2e^{x}-3e^{-x}=0$ is equivalent to the equation $\tanh(x)=-\frac{1}{5}$ and so solve the equation.

In Chapter 10 question 15 you learnt that by making the substitution $x=a\sin(u)$ you can show that $\int \frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\left(\frac{x}{a}\right)$.

25) By making the substitution $x=a\sinh(u)$, show that $\int \frac{1}{\sqrt{a^2+x^2}}dx=\text{arsinh}\left(\frac{x}{a}\right)+\text{C}$.

26 i) By making the substitution $x=a\cosh(u)$, show that $\int \frac{1}{\sqrt{x^2-a^2}}dx=\text{arcosh}\left(\frac{x}{a}\right)+\text{C}$.

ii) By making the substitution $x=a\tanh(u)$, show that $\int \frac{1}{\sqrt{a^2-x^2}}dx=\text{artanh}\left(\frac{x}{a}\right)+\text{C}$.