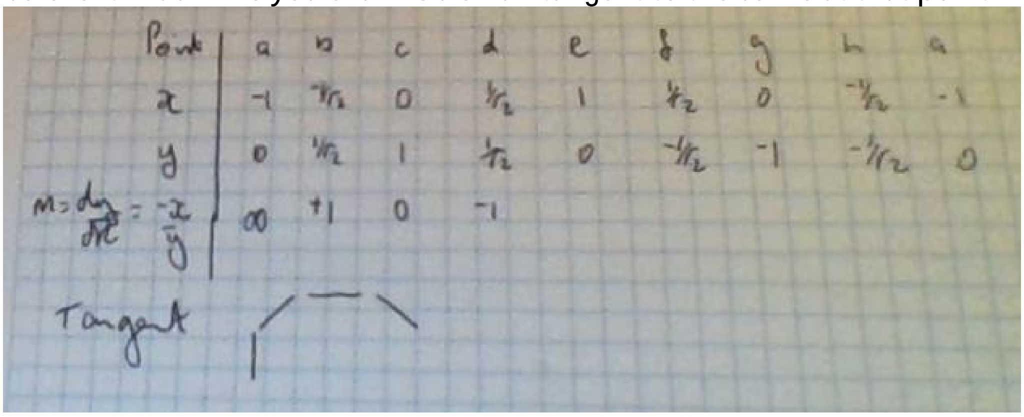
Chapter 15: Solving Differential Equations

1) If $\frac{dy}{dx} = 2x$, what is *y* as a function of *x*? If you said $y = x^2 + C$ then you have just solved a *differential equation*. Your answer was not just a curve $\left(y = x^2\right)$ but a *family of curves* $\left(y = x^2 + C\right)$, where C is any real number. Sketch the family of functions $y = x^2 + C$ for a range of values of C .
This set of curves is the *general solution* to the differential equation.

2) If you want to pick out which is the "true curve" from the family of curves you need some more information. If $\frac{dy}{dx} = 2x$ and $y = 3$ when $x = 2$, what is *y* as a function of *x*?
The "true curve" is called the *particular solution* to the differential equation.

3) Try this exercise. The table below is of a set of (*x*, *y*) points of some curve $y = f(x)$. We are not given the curve $f(x)$ but we are given the *gradient* of the curve $m = \frac{dy}{dx} = -\frac{x}{y}$, for each (*x*, *y*) point. Calculate each value of m and draw a short line with that gradient, the next line following directly from the one before it. Each line you draw is a small tangent to the curve at that point. An example is given below:



Point	a	b	c	d	e	f	g	h	a
x	-1	$-1/\sqrt{2}$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1
y	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0
$m = \frac{dy}{dx} = \frac{-x}{y}$	∞	$+1$	0	-1					
Tangent									

4) You have drawn a "tangent diagram" for the unknown curve. See if you can guess from the tangent diagram what the curve is.
i) Name the curve and give its equation.
ii) Show that the curve is the solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

5) i) Draw a similar table to the one above where $\frac{dy}{dx} = y$. Draw a tangent diagram and make a guess what the solution of the differential equation is.

ii) Repeat with the equation $\frac{dy}{dx} = x$.

iii) Repeat with the equation $\frac{dy}{dx} = \frac{1}{x}$.

6) You have solved four *differential equations* by drawing tangent diagrams as an approximation of the sketch of the curve
i) Show that all of the differential equations in questions 3 and 4 are of the form $\frac{dy}{dx} = \frac{g(x)}{f(y)}$, where $g(x)$ and $f(y)$ are certain functions, and hence the differential equation has the solution $\int f(y)\,dy = \int g(x)\,dx$.

ii) Create your own example of a differential equation in the form $\frac{dy}{dx} = \frac{g(x)}{f(y)}$ and solve it. You can always check your solution by differentiating.

The equations of the type in question 5 are known as "Separable First Order" differential equations. "Separable" means they can be separated into functions of just y on one side and just x on the other side. "First Order" means they involve only a first derivative $\frac{dy}{dx}$ and no higher order derivatives such as $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc.

6) i) Show (or at least convince yourself) that the equation $x^2\frac{dy}{dx} + 2xy = \sin(x)$ is *not* separable.

ii) Show that $x^2\frac{dy}{dx} + 2xy$ is the same as $\frac{d}{dx}(x^2y)$ and integrate the equation on both sides to show that it has the solution $x^2y = -\cos(x) + \mathrm{C}$.

The equation in question 6 is of the type known as an "exact equation", which is an equation that can be put into the form $\frac{d}{dx}f(x,y) = g(x)$. This has the solution $f(x,y) = \int g(x)dx$.

7) Show that $\frac{dy}{dx}\ln(x) + \frac{y}{x} = \frac{d}{dx}(y\ln(x))$. Hence solve the equation $\frac{dy}{dx}\ln(x) + \frac{y}{x} = x$.

8 i) Show (or at least convince yourself) that the equation $\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin(x)}{x^3}$ is *not separable* and is *not exact*.

ii) Show that the equation becomes an exact equation if you multiply through by x^3.

iii) Write the equation in the form $\frac{d}{dx}f(x,y) = \sin(x)$ and solve by integrating both sides.

If an equation is *not exact* you can sometimes *make it exact* by multiplying through by a certain function of x. The question is " which function of x ? ". In question 8 you may have been able to guess that multiplying through by x^3 would work. But it is not always easy to guess. In an exam you may be given the function, which is called an "integrating factor". But in a certain type of equation there is a trick available to find the right function.

8) Consider the equation $\frac{dy}{dx} - y = e^x$. Show that multiplying right through by e^x makes this equation *exact*. Solve the equation by integrating both sides. (Adding the "+ C " after the integral is important here.)

Note that the equation above is in the form $\frac{dy}{dx} + (-1) \times y = e^x$ and that $e^{\int -1dx} = e^{-x}$. This is a clue to how we find the right factor to multiply through. If the equation is in the form $\frac{dy}{dx} + \mathrm{P}(x)y = \mathrm{Q}(x)$ then we calculate the expression $e^{\int \mathrm{P}(x)dx}$ to give the integrating factor.

9) Show that if $\frac{dy}{dx}+\mathrm{P}(x)y=\mathrm{Q}(x)$ and $f(x)=e^{\int \mathrm{P}(x)dx}$ then $\frac{d}{dx}\left(f(x)y\right)=f(x)\frac{dy}{dx}+f(x)\mathrm{P}(x)y=f(x)\mathrm{Q}(x)$ and so $f(x)y=\int f(x)\mathrm{Q}(x)dx$.

10 i) $3x\frac{dy}{dx}+y=x$. Multiply this equation through by $\frac{1}{3x}$ to get $\frac{dy}{dx}+\mathrm{P}(x)y=\mathrm{Q}(x)$ and state the value of $\mathrm{P}(x)$ and $\mathrm{Q}(x)$.

ii) Set $f(x)=e^{\int \mathrm{P}(x)dx}$ and multiply through by $f(x)$ to get the equation $\frac{d}{dx}\left(f(x)y\right)=f(x)\mathrm{Q}(x)$, stating the functions $f(x)$ and $\mathrm{Q}(x)$.

iii) Hence show that $x^{\frac{1}{3}}y=\frac{1}{4}x^{\frac{4}{3}}+\mathrm{C}$ and complete the final solution of the equation for y in terms of x .

iv) Check your solution by substitutinging y and $\frac{dy}{dx}$ back into the original equation.

The solution with the parameter "+ C " is the *general solution* of the differential equation. The *particular solution* of the equation is the one where a particular point lies on the curve.

v) Find the *particular solution* to the equation above if the point (8,5) lies on the curve.

11) By making the substitution $u=1+x$, calculate the integral $\int x\sqrt{1+x}\,dx$.

This technique of substituting one variable for another can work in differential equations in the same way that it works for integrals. In differential equations, as in integrals, you are usually told the variable to use.

12) i) Show that the transformation $z=y^{\frac{1}{2}}$ transforms the equation $\frac{dy}{dx}-4y\tan(x)=2y^{\frac{1}{2}}$ into the equation $\frac{dz}{dx}-2z\tan(x)=1$.

ii) Show that by multiplying through by the factor $f(x)=e^{\int -2\tan(x)dx}$ this becomes an exact equation involving $\cos^2(x)$. (Note that the integral of $\tan(x)$ is $\ln|\sec(x)|$ and the derivative of $\tan(x)$ is $\sec(x)\tan(x)$.

iii) Hence show that $z\cos^2(x)=\int \cos^2(x)dx$ and integrate to solve the equation for z in terms of x.

iv) Recalling that $z=y^{\frac{1}{2}}$, substitute to get the final equation for y in terms of x .

Remember, there are two significant techniques displayed in this question. Firstly, if we have an equation of the form $\frac{dy}{dx}+\mathrm{P}(x)y=\mathrm{Q}(x)$ then we use an integrating factor of the form $f(x)=e^{\int \mathrm{P}(x)dx}$ to transform the equation to $\frac{d}{dx}\left(f(x)y\right)=f(x)\mathrm{Q}(x)$, and then it is easy. Secondly, if our equation is not in the exact form $\frac{dy}{dx}+\mathrm{P}(x)y=\mathrm{Q}(x)$, we can sometimes change it into that form with a careful choice of a new variable $z=g(x,y)$ (which will usually be given to you) and then we can use the integrating factor.

So far we have been looking only at *first order* differential equations - those that involve only y and $\frac{dy}{dx}$. We are going to look at a method to solve *second order* differential equations, but as a lead up to that we are going to solve some simple, separable first order equations in a particular way.

13) Show that each of these two differential equations is separable and solve the equations:

i) $3\frac{dy}{dx} - 2y = 0$

ii) $2\frac{dy}{dx} + 5y = 0$

14 i) Did you get $\mathrm{A}e^{\frac{2}{3}x}$ for the first equation and $\mathrm{A}e^{\frac{-5}{2}x}$ for the second? If so, can you see a pattern?

ii) If so, test your pattern by seeing if it works for this equation:- $4\frac{dy}{dx} + 3y = 0$. You can test your guess by substituting back in.

15) Show by substituting y and $\frac{dy}{dx}$ that the solution to the equation $a\frac{dy}{dx} + by = 0$ is $y = \mathrm{A}e^{-\frac{b}{a}x}$.

This works because e^x is the magic function which is its own derivative. It crops up everywhere, as you saw in Euler's formula for a complex number. These are "first order" differential equations involving only $\frac{dy}{dx}$. We are going to apply the same technique to "second order" differential equations that involve $\frac{d^2y}{dx^2}$, though we shall now use the short notation of y, y' and y''.

16) Consider the *second-order equation* $y'' + 5y' + 6y = 0$, where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

Let's make a *guess* that a solution to this equation is $y = \mathrm{A}e^{\alpha x} + \mathrm{B}e^{\beta x}$ for some constants α and β.

i) Show that by substituting $y = \mathrm{A}e^{\alpha x} + \mathrm{B}e^{\beta x}$ into the equation $y'' + 5y' + 6y = 0$ we get the equation $\left(\alpha^2 + 5\alpha + 6\right)\mathrm{A}e^{\alpha x} + \left(\beta^2 + 5\beta + 6\right)\mathrm{B}e^{\beta x} = 0$.

ii) Show that this requires that $\alpha^2 + 5\alpha + 6 = 0$ and that $\beta^2 + 5\beta + 6 = 0$ and conclude that α and β are the two solutions to the same quadratic.

iii) Solve the quadratic to find to α and β and substitute them into $y = \mathrm{A}e^{\alpha x} + \mathrm{B}e^{\beta x}$ to find the general solution to the second order differential equation .

iv) Substitute your solution for y back into the equation above to check that it works.

You have found that solving the differential equation requires that you first solve a quadratic equation. This quadratic equation is called the ***auxiliary equation*** of the differential equation.

17) Try to use the method above to solve the equation $y'' - 6y' + 9y = 0$.

Did you get $y = \mathrm{A}e^{3x} + \mathrm{B}e^{3x}$? That is because this quadratic equation has two equal roots and because A is an arbitrary constant, your two equal solutions are really one solution $y = \mathrm{A}e^{3x}$.

However there is another solution hiding here, and it is important with differential equations (or indeed any equations) that you include all possible solutions.

18) Show by substituting that $y = \mathrm{B}xe^{3x}$ is another solution to the equation in Question 17. Then show that the sum of the two solutions, $y = \mathrm{A}e^{3x} + \mathrm{B}xe^{3x}$, is also a solution to the equation .

All these differential equations are *linear* differential equations which means that if there are two solutions, namely $y = f(x)$ and $y = g(x)$, then $y = f(x) + g(x)$ is also a solution.

19) Show that if the equation $ay'' + by' + cy = 0$ has two independent solutions, $y = f(x)$ and $y = g(x)$, then $y = f(x) + g(x)$ is another solution.

20) Suppose that $y = \mathrm{A}xe^{\alpha x}$ is a solution to the equation $ay'' + by' + cy = 0$

i) Show that this gives you two equations involving α : $a\alpha^2 + b\alpha + c = 0$ and also $2a\alpha + b = 0$.
ii) Therefore show that *the auxiliary equation is a perfect square* with α as its double root.
iii) Show that $y = \mathrm{B}e^{\alpha x}$ is also a solution.
iv) Conclude that *in the case where the auxiliary equation is a perfect square*, then the solution is $y = \mathrm{A}xe^{\alpha x} + \mathrm{B}e^{\alpha x}$, where α is the double root.
v) Substitute $y = \mathrm{A}xe^{\alpha x} + \mathrm{B}e^{\alpha x}$ back into the equation $ay'' + by' + cy = 0$ and show that it works (when $a\alpha^2 + b\alpha + c = 0$ and $2a\alpha + b = 0$.)

It is nice in mathematics when you find unexpected connections between seemingly unrelated topics, such as differential equations and quadratic equations. And this happens more often than you might think. There are three cases of solutions to a quadratic equation - two real roots, one double root and no real roots. And you will know that “no real roots” means one pair of complex conjugate roots: $a + bi$ and $a - bi$.

21) Consider the equation $y'' + 4y' + 14y = 0$.
i) Show that the auxiliary equation has roots $2 + \sqrt{10}\,i$ and $2 - \sqrt{10}\,i$.
ii) Hence show that the general solution to the differential equation $y = \mathrm{A}e^{(2+\sqrt{10}\,i)x} + \mathrm{B}e^{(2-\sqrt{10}\,i)x}$ can be written as $e^{2x}\left(\mathrm{P}\cos\left(\sqrt{10}\,x\right)\right) + \mathrm{Q}\sin\left(\sqrt{10}\,x\right)$, where P and Q could be complex numbers.

You might be surprised to see a complex number coming up in the solution to the differential equation in question 21. But you should not be surprised really. You are quite used to complex solutions to a polynomial. P and Q can be any complex numbers in the general solution given. Whether the values of P and Q are real or imaginary will be determined when we have more information about the equation.

22) If I add the information that $y = 1$ when $x = 0$ and that $y' = 0$ when $x = 0$, show that in the solution above, P = 1 and Q = $-\frac{1}{5}\sqrt{10}$.

The extra information “ $y = 1$ when $x = 0$ and $y' = 0$ when $x = 0$ ” is called a “boundary condition” (unless time is the variable instead of x, when it is usually called an “initial condition”.) The solution to question 21 using general values P and Q is called the “general solution” to the equation. The solution to question 22 which uses extra information to find particular values for P and Q is called the “particular solution” to the equation. All the equations in questions 16 to 22 above have zero on the right side. What if they had a function $f(x)$ on the right hand side?

23) Consider the equation $y'' + 5y' + 6y = 3$.
i) Show that a proposed solution could be of the form $y = \mathrm{A}e^{-2x} + \mathrm{B}e^{-3x} + \lambda$ where λ is a real number.
ii) Substitute y back into the equation, calculating y'' and y' to show that $\lambda = \dfrac{1}{2}$.

24) Consider the equation $y'' + 5y' + 6y = 3x$ and a proposed solution of the form $y = \mathrm{A}e^{-2x} + \mathrm{B}e^{-3x} + \lambda + \mu x$ where λ and μ are real numbers. Show by substituting y back into the equation that $6\mu = 3$ and $5\mu + 6\lambda = 0$. So find λ and μ and give the final form of the solution.

When given an equation of the form $ay'' + by' + cy = f(x)$, the answers to the questions above suggest that you should first find a solution to the equation $ay'' + by' + cy = 0$ which is called the “complementary equation", and then add a form involving λ, μ etc like those above. The "form involving λ, μ etc" is called "particular integral".

25) Consider the equation $ay'' + by' + cy = f(x)$.

i) Show that the solution will be of the from $y = \mathrm{A}e^{\mathrm{p}x} + \mathrm{B}e^{\mathrm{q}x} + g(x)$, where p and q are the *roots* of the *complementary equation*.

ii) By substituting y back into the equation show that $ag''(x) + bg'(x) + cg(x) = f(x)$. Since this is an identity the function $g(x)$ may be found by giving it the same form as $f(x)$ and matching coefficients for equality.

iii) Given that $g(x) = \lambda$ in question 23 and $g(x) = \lambda + \mu x$ in question 24, suggest a form for $g(x)$ if $f(x)$ in question 25 is equal to $3x^2$.

iv) If $y'' + 5y' + 6y = 3x^2$ write general solution for y.

26) Consider the equation $y'' + 5y' + 6y = 6e^{3x}$.

i) Show that a proposed solution is of the form "y = *complementary function* + λe^{3x} " .

ii) Sub y into the equation and find λ.

27) Consider the equation $y'' + 5y' + 6y = 4\cos(2x)$. Show that the particular integral is $\lambda\cos(2x) + \mu\sin(2x)$ and find λ and μ .

28) $y'' - 4y' + 4y = 4e^{2x}$.

i) Find the *complementary function* for this equation and show that the *general solution* for the *complementary function* is $y = \mathrm{A}xe^{4x} + \mathrm{B}e^{4x}$. (Hint: Look again at question 20.)

ii) Explain why neither λe^{2x} nor λxe^{2x} can be a *particular integral* for this function.

iii) Show that $\lambda x^2 e^{2x}$ is a particular integral for this function. Find the value of λ and find the general solution of the equation.

iv) If $y = 10$ when $x = 0$ and $y' = 5$ when $x = 0$, find the *particular* solution to the equation. (in other words, find A and B.)