Chapter 16: Modelling with Differential Equations

In mathematics a "model" is an equation that describes a real situation. In this chapter we are going to model some physical processes with differential equations and analyse what the solutions to the differential equations tell us about the physical process. The models are principally problems of Mechanics which are dealt with in more detail in the Mechanics option of the Further Mathematics course.

Projectiles on the surface of the Earth

1) Jack and Jill went up the hill and looked over a cliff edge at the top down to the water below. (There was a rail for safety.) Jack leant out and dropped a stone *vertically* down to the water. Jill threw her stone straight out *horizontally*, released at the same time as Jack dropped his stone. Whose stone hit the water first?

I hope your answer to question 1 was "Both stones hit the water at exactly the same time." Jill's stone fell at the same rate as Jack's. The motion downwards was independent of the horizontal movement it was given. We are going to describe the movement of the stones with differential equations - one for the vertical motion and one for the horizontal. You need to know just three things: The cliff was 50 metres high, Jill threw her stone horizontally with an initial speed of 10 metres per second (ms^{-1}) and the acceleration of gravity on Earth is a constant g = 9.8 metres per second squared (ms^{-2}) .

2) Consider the vertical motion of both stones, ignoring any horizontal motion. *Acceleration* is the rate of change of *velocity* $\left(a = \frac{dv}{dt} = 9.8\,\text{m}\,s^{-2}\right)$. This is a (very simple) differential equation. Solve it to find the velocity as a function of time. To complete the function you will need to know the initial velocity of the stone. (The word "dropped" is a clue.)

3) *Velocity* is the rate of change of *displacement* $\left(v = \frac{ds}{dt}\right)$ where s is the change in position of the stone.

i) Solve this differential equation to find s, the displacement, as a function of t, time. (The initial displacement is zero at the top of the cliff.)

ii) Noting that the final displacement is 50m at the bottom of the cliff, show that the stone takes 3.19 seconds to reach the water.

These "equations of motion" are the same for Jack's stone as for Jill's stone. They describe identical motion *vertically*. However Jill's stone is also travelling *horizontally*. Her stone had an initial horizontal velocity of 10ms^{-1} but it had *no horizontal acceleration*. (We are ignoring air resistance, or any other resistance to motion, because we can.)

4 i) Convert the information above into two simple differential equations representing the *horizontal* motion of Jill's stone, showing that $\frac{dv}{dt}=0$ and $\frac{ds}{dt}=10\,\text{m}s^{-1}$.

ii) Noting that Jill's stone hits the water after 3.19 seconds of flight, calculate how far from the cliff it falls.

So far we have assumed the stone falls with constant acceleration, which is reasonable for a stone. But what if Jack and Jill dropped a *stick* off the cliff, instead of a stone. A stick is lighter than a stone and more likely to be affected by air resistance. We need a new "model" for acceleration to deal with this.

5) Consider this model for the acceleration of the falling stick: $a=\frac{g}{t+1}$, where g is Earth's gravity (9.8ms-2.)

i) Is this a *reasonable* model? Explain your answer.

ii) Noting that $a=\frac{dv}{dt}$, solve this equation to find velocity (v) as a function of time (t) and then displacement (s) as a function of time (t) .

iii) Find the velocity of the stone and the distance it has fallen after two seconds.

E 6) A reasonable question to ask is "How long does it take for the stone to hit the water?"

i) Show that this question is equivalent to the equation $g\left((t+1)\ln(t+1)-t\right)=50$.

ii) I don't know of any exact way to solve this equation but you may know of a numerical method to solve it, or you could solve it with a graphing app. Try this.

iii) Calculate the *speed* of the stone when it hits the water.

You might argue at this point that acceleration will be modelled more accurately in terms of *distance*, x , rather than *time*.

7) Consider this model for the acceleration of the falling stick: $a=g\left(1-\frac{x}{\text{K}}\right),\ \text{K}<50,\ 0\le x\le 50$

where K is a constant and x is the distance fallen.

i) Is this a reasonable model? Explain your answer.

The problem here is that we have $a=\frac{dv}{dt}=g\left(1-\frac{x}{\text{K}}\right)$. So the function is in terms of x but the derivative is in terms of t . It turns out this is easily fixed.

ii) Noting that $v=\frac{dx}{dt}$, show that $a=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

iii) By writing our model as $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=g\left(1-\frac{x}{\text{K}}\right)$, write the velocity, v , as a function of x .

iv) If K = 100m, calculate the velocity of the stick when it hits the water.

Projectiles in Space

A projectile on the Earth's surface falls with a constant acceleration g = 9.8 ms^{-2}. Isaac Newton showed us that the force of gravity on an object above the Earth is inversely proportional to distance from the Earth's centre.

8) Newton's law of gravity says that the force of gravity on object a distance x above the surface of the Earth is $\mathrm{F} = -\frac{\mathrm{k}\,m\,\mathrm{M}}{(\mathrm{R}+x)^2}$, where m is the mass of the object, M is the mass of the Earth, R is the radius of the Earth and k is a positive constant.

i) Find the value of k given that the force of gravity on an object on the Earth's surface is $\mathrm{F} = -\mathrm{m}g$ and write the force, F, in terms of g, not k .

ii) Show that the acceleration of the spacecraft is given by $a = -\frac{g\mathrm{R}^2}{(\mathrm{R}+x)^2}$, where x is its height above the Earth's surface.

iii) Suppose a spacecraft is falling directly towards Earth. At a certain distance X above the surface of the Earth it has speed U ms^{-1}.

Show that it's velocity towards Earth is given by $v^2 = \mathrm{U}^2 + 2g\mathrm{R}^2\left(\frac{1}{\mathrm{R}+x} - \frac{1}{\mathrm{R}+X}\right)$.

iv) If the spacecraft was one Earth radius above the Earth, so that X = R, calculate its speed when it hits the Earth.

Springs and other oscillating particles

Now let's consider a *spring*, lying horizontally on a frictionless table. One end is fixed to a point and cannot move. The other end is attached to a particle of mass m kg. You move the particle a distance $x = a$ metres from its resting position, which is at $x = 0$. You stretch the spring and then let it go. Describe what happens next.

This "back-and-forward" process is called "oscillation". In reality, friction will do its work and the oscillation will eventually run out of energy and stop. If friction is high it may stop before it gets a chance to oscillate. But we imagine a perfect spring and a smooth table where friction is zero.

9 i) Consider the equation $\ddot{x} = -\omega^2 x$, where ω is a constaant. (" $\dot{x}$ " is a shorthand way of writing $\frac{dx}{dt}$ and " $\ddot{x}$ " is a shorthand way of writing $\frac{d^2x}{dt^2}$. It is a notation invented by Isaac Newton.)

Consider the places when $\ddot{x}$ is positive, when it is negative and when it is zero and explain why the equation adequately models the movement. (Physicists will recognise the equation $\ddot{x} = -\omega^2 x$ as *Hooke's Law*.)

ii) Noting that $\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ show that $v^2 = -\omega^2 x^2 + \mathrm{C}$, where c is a constant of integration.

iii) Noting that that the mass was at rest when you held it at $x = a$, show that $v^2 = \omega^2\left(a^2 - x^2\right)$.

iv) Show that the velocity is maximum when the acceleration is zero.

10) The equation in question 9, $\ddot{x} = -\omega^2 x$, models the movement of an oscillating spring.

i) Show that the equation may be written as $\ddot{x} + \omega^2 x = 0$ and that the differential equation may be solved by the methods learnt in Chapter 15 to give the general solution $x = \mathrm{A}e^{i\omega t} + \mathrm{B}e^{-i\omega t}$.

ii) Show that the "initial conditions" of the spring as described in question 9 tell us that $x(0) = a$ and $\dot{x}(0) = 0$. Hence find the values of A and B.

iii) Show that the model of the movement may be written as $x = a\cos(\omega t)$.

11) A mass of 2kg is attached to a spring, lying horizontally on a frictionless table. When the mass is pulled 60 cm away to the right from its unstretched position it feels a force generated by the spring equal to 3 Newtons.

i) Use Newton's Law $(\mathrm{F} = m\ddot{x})$ and Hooke's Law $(\mathrm{F} = -\mathrm{k}\,x)$ to find k and so model the oscillation of the spring with the formula $\ddot{x} = -\omega^2 x$, stating the value of ω^2 .

ii) Write a formula relating the *velocity* of the mass to its *position*.

iii) Calculate the maximum speed of the particle during one cycle.

iv) Calculate the speed of the particle when it is 10cm to the right of the resting position.

v) Sketch a graph of velocity (v) against displacement (x).

vi) Write a formula relating the *position* of the mass (x) in terms of *time* (t).

vii) Calculate where the particle is, in relation to the resting position, five seconds after it is let go.

viii) The "period" of the motion is the time it takes to return to its starting point, so completing one cycle. Show that if the position x is modelled with $x = a\cos(\omega t)$ the period of the motion is $\frac{2\pi}{\omega}$ and state the period of this motion accurate to three significant figures.

ix) Sketch a graph of displacement (x) versus time (t) .

Our "oscillator model" has so far ignored the effect of friction. This makes the problem simpler but it is not terribly realistic. Friction is always present and sometimes it can be quite significant. Friction will always oppose motion and is usually proportional to speed. A resistive force that always opposes motion is sometimes called a "damping effect" and an oscillator model that accounts for friction is called a "damped oscillator". The model for a damped oscillator can then be written as $\mathrm{F}=\mathrm{F}_{spr}+\mathrm{F}_{res}=-\mathrm{k}x-\mathrm{b}\dot{x}$. The total force (F) is composed of the force provided by the spring (or other oscillating mechanism) $\mathrm{F}_{\mathrm{spr}}$ plus the resistive force $\mathrm{F}_{\mathrm{res}}$. "k" and "b" in the equation are always positive constants.

12 i) Explain why $\mathrm{F}_{res}=-\mathrm{b}\dot{x}$ is a reasonable model of a resistive force, such as friction.

ii) Show that oscillator model above is equivalent to the differential equation $m\ddot{x}+\mathrm{b}\dot{x}+\mathrm{k}x=0$, where m is the mass of the oscillating particle.

iii) Show that this equation has the general solution $x=\mathrm{A}e^{\lambda_1 x}+\mathrm{B}e^{\lambda_2 x}$, where

$$\lambda_1=\frac{-\mathrm{b}+\sqrt{\mathrm{b}^2-4\mathrm{k}m}}{2m} \text{ and } \lambda_2=\frac{-\mathrm{b}-\sqrt{\mathrm{b}^2-4\mathrm{k}m}}{2m}.$$

iv) Explain how the values of λ_1 and λ_2 depend on the relative magnitudes of b^2 and $4\mathrm{k}m$.

v) Your explanation should have divided the values of λ_1 and λ_2 into three separate cases. Return, if necessary, to chapter 15 to remind yourself how the three cases relate to three different forms of the general solution to the equation $x=\mathrm{A}e^{\lambda_1 x}+\mathrm{B}e^{\lambda_2 x}$.

13) Return to the model in question 11 where $\mathrm{F}_{spr}=-5x$. This time we add a resisting force of friction $\mathrm{F}_{res}=-2\dot{x}$ so that $\mathrm{F}=\mathrm{F}_{spr}+\mathrm{F}_{res}=-5x-2\dot{x}$.

i) Since we still have a particle of mass 2 kg attached to the spring , show that the acceleration of the particle is modelled as $\ddot{x}=-\frac{5}{2}x-2\dot{x}$ and that this gives us the differential equation $2\ddot{x}+4\dot{x}+5x=0$.

ii) Show that this equation has the general solution $x=e^{-t}\left(\mathrm{A}e^{i\lambda t}+\mathrm{B}e^{-i\lambda t}\right)$ where λ is the imaginary part of the roots of the characteristic equation.

iii) Show that this can be written as $x=e^{-t}\left((\mathrm{A}+\mathrm{B})\cos(\lambda t)+(\mathrm{A}-\mathrm{B})\sin(\lambda t)\right)$ and then use the initial conditions $x(0)=0.6$ and $\dot{x}(0)=0$ to show that $x=0.6e^{-t}\left(\cos(\lambda t)+\frac{1}{\lambda}\sin(\lambda t)\right)$ and that this in turn may be written as $x=0.6\mathrm{P}e^{-t}\cos(\lambda t+\mathrm{Q})$ where $\mathrm{P}\cos\mathrm{Q}=1$ and $\mathrm{P}\sin\mathrm{Q}=-\dfrac{1}{\lambda}$.

iv) Check that when $x=0.6\mathrm{P}e^{-t}\cos(\lambda t+\mathrm{Q})$ then $x(0)=0.6$ and $\dot{x}(0)=0$.

v) Check that when $x=0.6\mathrm{P}e^{-t}\cos(\lambda t+\mathrm{Q})$ then $2\ddot{x}+4\dot{x}+5x=0$. $\left(\mathrm{Hint}: \lambda^2=\dfrac{3}{2}\right)$

vi) Show that $\mathrm{P}=\sqrt{\frac{5}{3}}$ and $\mathrm{Q}=\arctan\left(-\frac{1}{3}\sqrt{6}\right)$.

vii) Sketch a graph of $x=0.6\mathrm{P}e^{-t}\cos(\lambda t+\mathrm{Q})$ for those values of P and Q.

Your sketch of $x = 0.6\,\mathrm{P}e^{-t}\cos(\lambda t + \mathrm{Q})$ should have a shown a repeated cosine curve with reducing amplitude. This models an oscillating particle whose oscillations get nearer and nearer to zero. The reduction in amplitude is caused by friction, which in this case had been set at two times the velocity. Let's double the strength of friction and see what happens.

14)Return to our model $\mathrm{F} = \mathrm{F}_{spr} + \mathrm{F}_{res} = -\mathrm{k}x - \mathrm{b}\dot{x}$, but this time our parameter b , representing the strength of friction , will be 4 instead of 2 .
i) Show that this gives us the differential equation $2\ddot{x} + 8\dot{x} + 5x = 0$.
ii) Show that this equation has the general solution $x = \mathrm{A}e^{\lambda_1 t} + \mathrm{B}e^{\lambda_2 t}$ where λ_1 and λ_2 are the distinct real roots of the characteristic equation.
iii) Use the initial conditions $x(0) = 0.6$ and $\dot{x}(0) = 0$ to find A and B.
iv) Check your solution by substituting x, $\dot{x}$ and $\ddot{x}$ into the differential equation.
v) Sketch a graph of $x = \mathrm{A}e^{\lambda_1 t} + \mathrm{B}e^{\lambda_2 t}$ for those values of A and B.

Your sketch should have shown an exponential curve starting at x = 0.6 when t = 0 and decreasing towards x = 0. Clearly this is a very different situation than the oscillating particle of question 11. The difference is caused by the sign of the discriminant of the characteristic equation - negative in question 11, positive in question 14. Between these two there is the case when the discriminant of the characteristic equation is zero.

15 i) If $\mathrm{m}\ddot{x} + \mathrm{b}\dot{x} + \mathrm{k}x = 0$, show that the discriminant of the characteristic equation is zero when b^2 equals $4\mathrm{km}$.
ii) When $2\ddot{x} + \mathrm{b}\dot{x} + 5x = 0$, find the value of b such that the discriminant of the characteristic equation is zero.
iii) Solve the equation $2\ddot{x} + \mathrm{b}\dot{x} + 5x = 0$ for this value of b.
The solution should be in the form $e^{\lambda t}(\mathrm{A} + \mathrm{B}t)$.
iv) Use the initial conditions $x(0) = 0.6$ and $\dot{x}(0) = 0$ to find A and B.
v) Check your solution by substituting x, $\dot{x}$ and $\ddot{x}$ into the differential equation.
vi) Sketch a graph of $x = e^{\lambda t}(\mathrm{A} + \mathrm{B}t)$ for those values of A and B and compare it for with the graph in question 14.

Predator and Prey models

In New Zealand, where I come from, they have a big problems with rabbits. The rabbits were introduced by the British colonial settlers, presumably to remind them of home, and the rabbits found a comfortable home in New Zealand. There was plenty of empty space, plenty of grass to eat and no animals that wanted to eat them. The rabbits multiplied rapidly until they were eating pasture that farmers wanted for their sheep and cows. The farmers thought something should be done about this and somebody suggested introducing foxes from Britain so that the foxes would prey on the rabbits. Think about that for a moment and say why it was a very bad idea.

Of course the foxes would thrive like the rabbits did. They had plenty of rabbits to eat and nothing that wanted to eat them. In time the rabbit numbers would decline and then the fox numbers would decline because they had less rabbits to eat. And as the fox numbers declined the rabbits would start to increase and the cycle would repeat itself.

The foxes who eat rabbits are called "predators" and the rabbits who get eaten are called "prey". We have mathematical models, called "predator-prey models" that can predict with surprising accuracy the changing numbers of predators and preys. They require two differential equations, one for the predators and one for the prey and these depend on each other. So they must be solved simultaneously, rather like simultaneous algebraic equations.

16) Let us first consider rabbits by themselves. We assume the population grows "exponentially", which means in proportion to its current size. So if *r* represents the number of rabbits at a time *t* then the rate of increase in number is modeled by $\frac{dr}{dt} = \mathrm{a}r$, where *a* is a positive constant.

i) Show that this gives us the equation $r = \mathrm{A}e^{\mathrm{a}t}$.

ii) *t* is measured in weeks. 100 rabbits were introduced at the start and they were found to double their number every 10 weeks. Find the parameters A and a .

iii) How many rabbits were there after 100 weeks? (If you know your powers of two you can do this quickly in your head.)

17) Now we bring in foxes. We assume four things:

- In the absence of foxes the rabbit numbers increase at a rate equal to the current population times $\mathrm{a} = 0.0693$.
- Fox numbers increase at a rate equal to one tenth of the number of rabbits.
- On average each fox kills one rabbit per week.
- At the start there were 100 rabbits and 10 foxes.

Use *r* for the number of rabbits at time *t* and *f* for the number of foxes.

i) Use the first 3 assumptions to write 2 differential equations: $\dot{f} = 0.1r$ and $\dot{r} = \mathrm{a}r - f$ $(\mathrm{a} = 0.0693)$

ii) Differentiate the second equation and substitute in the first equation to get the second order linear equation $\ddot{r} - \mathrm{a}\dot{r} + 0.1r = 0$.

iii) Show that this equation has a solution of the form $r = e^{\lambda t}\left(\mathrm{A}\cos(\mu t) + \mathrm{B}\sin(\mu t)\right)$, where $\lambda = \frac{\mathrm{a}}{2}, \mu = \frac{1}{2}\sqrt{0.4 - \mathrm{a}^2}$, $\mathrm{a} = 0.0693$ and for simplicity we can take $\mu = \sqrt{0.1}$.

iv) Show that the initial conditions tell us that $r(0) = 100$ and $\dot{r}(0) = -3.06$

and use this to find A and B.

v) Show that $r = e^{\lambda t}\left(\mathrm{A}\cos(\mu t) + \mathrm{B}\sin(\mu t)\right)$ may be written as $r = \mathrm{P}e^{\lambda t}\cos(\mu t + \mathrm{Q})$ and find P and Q.

vi) Noting that $\dot{f} = 0.1r = 10.2e^{\lambda t}\cos(\mu t + 0.204)$ show that

$f = \frac{10.2}{\lambda^2 + \mu^2} e^{\lambda t}\left(\lambda\cos(\mu t + 0.204) + \mu\sin(\mu t + 0.204)\right) + \mathrm{C}$ and that this may be written as

$f = \mathrm{S}e^{\lambda t}\cos(\mu t + \mathrm{T}) + \mathrm{C}$, finding the values of S and T.

vii) Since there were 10 foxes at the start, find C .

viii) Sketch comparison graphs of the population of rabbits and of foxes over time.

ix) Look at your graphs and consider whether they are a good model of our rabbits and foxes.

You will probably first notice that the curve oscillates about zero, with a negative population below the line. This can be fixed by raising each curve by an amount equal to an estimate of the average population size. But even then you will see that both curves will always drop below zero at some point, indicating that eventual extinction of both foxes and rabbits is inevitable, which is not an unreasonable scenario.

That rather gloomy conclusion brings us to the end of the Further Mathematics syllabus. Congratulations. You have finished. However, there is a good deal more to these predator-prey models than what has just been touched on here. The interested reader might like to investigate the Lotka-Volterra equations, devised in the 1920's, independently, by the US mathematician Alfred Lotka and the Italian mathematician and physicist Vito Volterra. These equations were not solvable with neat final equations as our simplified version was. They made use of methods such as the time-independent "phase space" to examine the interactions that happen when we try to model a complex world. Albert Einstein said "The most incomprehensible thing about the world is that it *is* comprehensible."