Chapter 2: Complex Numbers

1) Solve these equations:

   i) $x^2 - 5x + 6 = 0$

   ii) $3x^2 + 2x - 5 = 0$

   iii) $x^2 + 2x + 10 = 0$

2) You might have had some trouble with question 1 iii). Did you write "No solutions" or "No real solutions"? *If so, I want you to imagine that there is a number, let's call it "i", that has the property that* $i^2 = -1$. The number "1" is a solution to (or "root of") the equation $x^2 - 1 = 0$. What equation is the number "*i*" a solution to (or root of)?

3 i) If $i^2 = -1$ what does i equal?
ii) The number 1 has two roots. What are they?
iii) Does the number -1 have two roots? If so, what are they?

4) Question 1.iii required you to take the square root of -9. If you write -9 as $9i^2$ you can probably work out its square root and use this to give two solutions to equation 1.iii.

5) Now solve these equations: (make sure you give both solutions.)

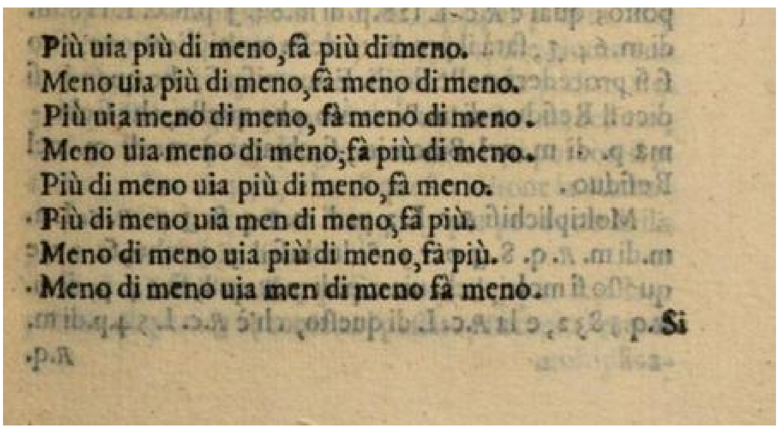
   i) $x^2 - 2x + 2 = 0$

   ii) $x^2 + 6x + 25 = 0$

6 i) You will have noticed that the two roots for each equation above come in matching pairs. What is the same about each of the roots in the pair? *(This is an important point that we will refer to in other questions.)*
ii) How do you think you would add two complex numbers? How do you think you would multiply them?
iii) Something special happens when you add the two roots of an equation, for example $(1+i)$ and $(1-i)$, and also when you multiply them. What is it that happens?

7) You should have a calculator with a "complex" mode which will operate on complex numbers. It will also have a key for the number *i*. Put your calculator into complex mode and repeat the operations in question 6 on your calculator. (If you haven't got a calculator with a complex mode then get one.)

8) Pick one root from equation 5.i, for example $(1+i)$, and one root from equation 5.ii, for example $(-3+4i)$. Add them by adding real parts and adding complex parts. Multiply them by multiplying out the brackets as $(a+bi)(c+di)$ and noting, of course, that $i^2 = -1$. Check your results on your calculator. Repeat for the four combinations of roots.


Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

In 1572 the Italian Mathematician Rafael Bombelli published his famous work "L'Algebra".

On the left is a page from Bombelli's book in which he describes the rules for complex number multiplication. Bombelli was not sure what $\sqrt{-1}$ was. He called the new number "plus of minus". We now know it as the complex number *i*. But he proved himself to be an excellent mathematician by ignoring what it *was* and figuring out what it *did*. He clearly stated consistent rules for arithmetic with this new number. The first of the eight rules pictured translates as "plus times plus of minus equals plus of minus" or, as we should say $1\times i=i$. The fifth line says "plus of minus times plus of minus equals minus" or, as we would say $i^2=-1$, a breakthrough indeed.

R E 8) Translate the eight lines of Bombelli's statement of the properties of $\sqrt{-1}$.

9) Any complex number may be written as $a+bi$, where a and b are "real" numbers. *(Note that this does not mean there is anything "unreal" about complex numbers.)* We often (but not always) use the letter "z" to refer to complex numbers and "x" and "y" to refer to real numbers. Add them by adding real parts and adding complex parts. Multiply them by multiplying out the brackets as $(a+bi)(c+di)$ and noting, of course, that $i^2=-1$.

i) If $z_1=a+bi$ and $z_2=c+di$ then work out z_1+z_2 and $z_1\times z_2$.

ii) If $z_1=a+bi$ and $z_1=a-bi$ then work out z_1+z_2 and $z_1\times z_2$.

10) If $z_1=a+bi$ then the "complex conjugate" of z is $z=a-bi$ written $\overline{z}$. If $z_1=a+bi$ work out, in terms of a and b, $z+\overline{z}$ and $z\overline{z}$. What is always true about $z+\overline{z}$ and $z\overline{z}$?

11) In question 6 we asked you what was the same about each pair of roots. Can you answer this question with a bit more detail now?

12 i) Write in terms of p and q the two roots to the equation $x^2+px+q=0$ where p and q are real numbers. (Complete the square or use the formula.)

ii) State the condition by which the two roots are either real and distinct, real and equal or a complex conjugate pair.

iii) Assuming the roots are a complex conjugate pair, call them z and $\overline{z}$. Work out $z+\overline{z}$ and $z\overline{z}$.

Show that the expression $x^2+px+q=0$ can also be written as $(z-\overline{z})(z+\overline{z})$.

iv) Can any quadratic always be written in this way, as a product of x minus a root?

E 13) If $(a+bi)$, b not equal to zero, is one root of the equation $x^2+px+q=0$ show that the other root *must* be $(a-bi)$. *Conclude that there are always two roots of every quadratic and that they are either two real roots (which may not be distinct) or else a complex conjugate pair.*

14) The roots of the quadratic equation $x^2+8x+25=0$ are α and β.

Find i) α and β ii) $\alpha+\beta$ iii) $\alpha\beta$

15) i) Find the quadratic equation that has roots $(-5+4i)$ and $(-5-4i)$.

ii) If $(3-5i)$ is one root of a quadratic equation with real coefficients, write down the quadratic equation.

16) In question 8 you added and multiplied the numbers $(1+i)$ and $(-3+4i)$, together with their complex conjugates. I am sure you have also figured out how to subtract them. Can you see how to *divide* $(-3+4i)$ by $(1+i)$?

Hint: We divide 2 by 3 by multiplying 2 by ⅓ and getting ⅔. So we could divide $(-3+4i)$ by $(1+i)$ by multiplying $(-3+4i)$ by $\dfrac{1}{(1+i)}$, but that raises the question "what is $\dfrac{1}{(1+i)}$?".

17 i) The answer to the previous question might be prompted by recalling how you did arithmetic with "surds" - numbers with irreducible square roots . What is $\frac{1}{1+\sqrt{2}}$ *? Is it the same as* $\frac{1}{1+\sqrt{2}}\times\frac{1-\sqrt{2}}{1-\sqrt{2}}$ *? And what does* $\frac{1}{1+\sqrt{2}}\times\frac{1-\sqrt{2}}{1-\sqrt{2}}$ equal?

ii) What is $\frac{-3+4\sqrt{2}}{1+\sqrt{2}}$?

iii) $1+\sqrt{2}$ and $1-\sqrt{2}$ are sometimes called "conjugate surds". If x_1 and x_2 are conjugate surds what you can say about x_1+x_2 and x_1x_2 ?

iv) If z and $\bar{z}$ are *complex conjugates* what you can say about $z+\bar{z}$ and $z\bar{z}$?

v) In the same way that you "rationalise the denominator" of a surd, you must "real-ise" the denominator of a complex number - make it real. Look at the fraction $\frac{1}{(1+i)}$ and multiply on top and bottom by the complex conjugate of $(1+i)$. Then work out $\frac{(-3+4i)}{(1+i)}$.

vii) Work out $\frac{(-3+4i)}{(1-i)}$, $\frac{1}{(-3+4i)}$, $\frac{(1+i)}{(-3+4i)}$, $\frac{(1-i)}{(-3-4i)}$. Check the results on your calculator.

The complex number $a+bi$ has a "real part" (a) and an "imaginary part" (b*i*) . These are kept separate in addition so that, for example, the sum of "2" and "3*i*" is $2+3i$ without further simplification. This suggested to some mathematicians of the eighteenth century that the real part and the imaginary part could be represented on separate axes and the complex number would become a point on the x-y plane, rather than a number on the real number line.


ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

LES QUANTITÉS IMAGINAIRES

DANS

LES CONSTRUCTIONS GÉOMÉTRIQUES.

This is the frontispiece of a small book published by Jean-Robert Argand, in French, in which he invents the Argand diagram. You may be able to find a copy in English or your own language to read or browse through

18 i) Draw ordinary x-y axes on squared paper. Mark the y axis as "Imaginary" and the x axis as "Real". The roots to the equation in Question 14 are $(4+3i)$ and $(4-3i)$. Plot these as points on your axes with coordinates (4,3) and (4,-3).

ii) Plot the roots to the equations in Question 5 in the same way. What can you say about the two roots of a quadratic equation when plotted on x-y axes ?

19) Draw an arrow from the origin to the point at (4,3). In vector terms, this is the "position vector" for the point at (4,3) on the axes. When the vector of a complex number is drawn on "Imaginary" and "Real" axes this is called an "Argand diagram". Draw the Argand diagram for the number $(1+2i)$. Copy the same arrow you just drew but start it at the point of the Argand diagram for $(4+3i)$. The arrow should finish at coordinates (5,5). Show that the resultant vector" is the vector from the origin to (5,5). Show that this arrow is the Argand diagram for the sum of vectors $(4+3i)$ and $(1+2i)$.

20) On separate axes draw Argand diagrams to show the sum of $(1+2i)$ and $(3-i)$. Check your results with arithmetic.

21) Draw the Argand diagram for $(4+3i)$. Draw the vertical line to make a right-angled triangle with *x*-axis. What is the length of the horizontal and vertical sides of the triangle? What is the length of hypotenuse - i.e. of the arrow itself? This length is called the "modulus" or the "magnitude" of the number $(4+3i)$ and is written $|4+3i|$. Find $|1+2i|$ and $|3-i|$.

22) Return to the right-angled triangle made by the Argand diagram for $(4+3i)$. Using trigonometry, what is the angle between the diagram and the *x* axis? (In radians - the angle with complex numbers must always be in radians.) This angle is called the "argument" of the complex number $(4+3i)$, and can be written as $\arg(4+3i)$. Calculate the argument of $(1+2i)$ and $(3-i)$.

Your calculator should have a "modulus" and "argument" function when in complex mode. It should also be able to switch a number from $a+bi$ form to "modulus and argument." Use this to check your answers on your calculator.

23 i) Draw the Argand diagram for $(4+3i)$. Using trigonometry, express the horizontal side, 4, in terms of the modulus and argument. Express the vertical side, 3, in terms of the modulus and argument. Express the number $(4+3i)$ in terms of its modulus and argument.

ii) Write each of the numbers $(1+2i)$ and $(2-i)$ in "modulus, argument" form.

24) If $z=a+bi$, state in terms of a and b the modulus and argument of z.

25) The modulus of z is written as Mod(z) or $|z|$ and is usually given the letter "r". The argument of z is written as arg(z) and is usually given the Greek letter "θ". Write z in "modulus and argument" using r and θ. This form is sometimes abbreviated to $r\ cis\ \theta$. What does the "cis" stand for in this case?

26) These numbers are written in $r\ cis\ \theta$ form. Write them in $a+bi$ form:

i) $6\ cis\ \frac{3\pi}{4}$ ii) $\sqrt{3}\ cis\ \frac{\pi}{3}$

27) Mark all these points on an Argand diagram: $(1+2i)$, $(2-i)$, $-(2+i)$, $(-1+2i)$.

Do they all lie on a circle?

What is the radius of the circle?

What is the modulus of all these numbers?

28) Note that for all real numbers x and y $|xy|=|x||y|$.

i) By writing $z_1=r\ cis\ \theta$ and $z_2=r\ cis\ \phi$ show that $|z_1z_2|=|z_1||z_2|$.

ii) By writing $z_1=a+bi$ and $z_2=c+di$ show that $|z_1z_2|=|z_1||z_2|$.

iii) If $z_1=-1+2i$ and $z_2=4+2i$ verify that $|z_1z_2|=|z_1||z_2|$.

You have solved many equations in real numbers. We are going to solve equations in complex numbers. The solution to your equation will always be a complex number. Even if it comes out as a real number that still counts because the complex numbers include the real numbers. A number is called “imaginary” if it is equal to ki where k is real.

29 i) If $z_1 = a + bi$ and z is real what can you say about b?

ii) If $z_1 = a + bi$ and z is imaginary what can you say about a?

30) Solve these equations for the complex number x:

i) $3z = (3+2i)$, ii) $z + 3 = 5 - i$, iii) $2z - 3 = 5$, iv) $2z - 4i = 6$, v) $2z - 4i = 6i$,

vi) $2z + 4i = 4z$, vii) $2z - (3+4i) = 6z + (2-i)$, viii) $\frac{1}{z} = (3-4i)$, ix) $\frac{1}{z+2} = (3+4i)$.

31 i) If z is the “square root” of $(3+4i)$ what can you say about z^2 ?

ii) If $z = a + bi$, what does z^2 equal in terms of a and b ?

iii) If $z^2 = (3+4i)$ write out and solve two equations involving a and b and the numbers 3 and 4. (Note that a and b are *real numbers*.)

iv) If $z^2 = (3+4i)$ what does z equal?

v) What is the square root of $(3+4i)$?

vi) Did you give just one answer or two? If you gave one answer find another one.

vii) Check your answers by squaring them and getting the answer $(3+4i)$

32) Find the square roots of these numbers:

i) $(2-5i)$ ii) $(5+12i)$ iii) $(1+i)$ iv) i

Check your answers by squaring them

33) In Question 13 you showed that that there are always two roots of every quadratic and that they are either two real roots (which may not be distinct) or else a complex conjugate pair.

Show that if $f(x) = x^2 + px + q = 0$ where p and q are real numbers, the quadratic may be written as

$f(x) = (x - (a+b))(x - (a-b))$ if $p^2 \geq 4q$ and

as $f(x) = (x - (a+bi))(x - (a-bi))$ if $p^2 < 4q$. In each case write a and b as functions of p and q.

34) Let $f(x)$ be the cubic function $2x^3 - 4x^2 - 5x - 3$.

i) Show that $x = 3$ is a root of $f(x)$.

ii) Show that $f(x)$ may be written as $(x-3)g(x)$ where $g(x)$ is a quadratic function.

iii) Find the two roots of $g(x)$ and write $f(x)$ as a product of three factors.

A “polynomial” is a function of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots .. + a_1 x + a_0$. The “degree” of the polynomial is the value of “n” in the expression above. A “quadratic” is a polynomial of degree 2. A “cubic” is a polynomial of degree 3. A “quartic” is a polynomial of degree 4. The “Fundamental Theorem of Algebra” says that every polynomial of degree n has exactly n roots.

iv) Recall that in a quadratic equation with real coefficients the two roots will always be both real, or a complex conjugate pair $a + bi,\ a - bi$. Explain why in a cubic equation there will always be three real roots or one real root and a complex conjugate pair.

35) The cubic $f(x) = x^3 + 4x^2 - 15x - 68$ has $(-4+i)$ as one of its roots. Find the other two roots and factorise $f(x)$.

36) Explain why a quartic equation with real coefficients will always have either two or four or no real roots.

37) The quartic $f(x) = x^4 + 2x^3 - x^2 + 38x + 130$ has $(2+3i)$ as one of its roots. Find the other three roots and factorise $f(x)$.

38) i) Find the three solutions to the equation $z^3 - 1 = 0$ and plot your solutions on an Argand diagram.

ii) Find the four solutions to the equation $z^4 - 1 = 0$ and plot your solutions on an Argand diagram.

iii) By observing the pattern of the Argand diagrams, plot what you think are the eight solutions to the equation $z^8 - 1 = 0$ on an Argand diagram.

iv) Write the solutions in the modulus/argument form $r\,cis\,\theta$ and in the form $a + bi$. Check your answers by raising them to the eighth power. (Hint: $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$)

39) Use the polar coordinates graph on the left to plot your roots in $r\,cis\,\theta$ form.

i) Find the two solutions to the equation $z^2 - 1 = 0$ in $r\,cis\,\theta$ form and plot your solutions on an Argand diagram.

ii) Find the four solutions to the equation $z^4 - i = 0$ and plot your solutions on an Argand diagram.

iii) By observing the pattern of the Argand diagrams, plot what you think are the eight solutions to the equation $z^8 - 1 = 0$ on an Argand diagram.