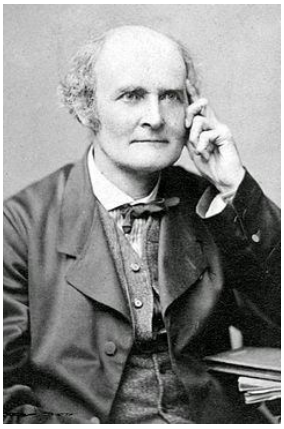
Chapter 3: Matrices I



R) Arthur Cayley (1821 - 1895) (left) and James Sylvester (1814-1897) (right) were two Victorian mathematicians who were almost exactly contemporary in time and place. They were close friends who lived remarkably similar and parallel lives and together they more or less invented matrix algebra. Their lives make an interesting read if you want to look them up.

1) Solve this simultaneous equation in two variables:

$$2x + y = 8$$
$$x - y = 1$$

You could also write this equation in terms of vectors, like this: $\begin{pmatrix} 2x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

Or as a matrix equation like this: $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

This demonstrates the way we multiply matrices.

2) Follow the same method for multiplication to calculate $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$.

3) What property of the rows and columns in these matrices is required so that we may multiply them?

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

4) Which of these matrices can be multiplied together and which cannot? Why?"

5) Multiply together those matrices in question 3 which can be multiplied. *(Note that we use AB to mean the product of matrices A and B.)*

Note that your calculator should have a "Matrix" function that will do matrix arithmetic for you. (If it it doesn't then get one that does. You will need to learn how to do matrix arithmetic without a calculator, but you can always use your calculator to check your answers.)

6) We can take the matrix product AC but not the matrix product CA. What does this tell you about matrix multiplication compared to number multiplication?

The "Dimension" of a matrix is m x n, where m is the number of rows and n is the number of columns.

8) i) Write the dimension of matrices A, B, C and D.

ii) By noting the dimensions of the matrices which can be multiplied and those which cannot, complete this sentence:

"If matrix A is m x n and matrix B is p x q we can multiply AB if and only if ....,
And we can multiply BA if and only if ...."

Adding and Subtracting matrices is done in the "obvious" way - by adding or subtracting each element of the matrix.

9) If $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, work out A + B, A - B, 2A + 3B.

10) Look at the four matrices in question 3. Can you add or subtract them? Why or why not?

Before looking at the properties of addition and multiplication of matrices let us look at the properties of addition and multiplication of numbers. Let a, b, c, d be any numbers.
The five statements written below are true for all numbers a and b. *Note that we use small letters for numbers and capital letters for matrices.*
i) a + b = b + a and ab = ba
ii) (a + b) + c = a + (b + c) and (ab)c = a(bc)
iii) a(b + c) = ab + ac
iv) For every number a there is another number b such that a + b = 0;
v) For every number a except 0 there is another number b such that ab = 1

11) Write the statements above but as applied to matrices A, B, C and D. If you think the statements are true, then leave them as they are. If you think they are false, say so and give a counterexample.

Note: Rule number 1 is called "the commutative property of addition and multiplication".
Rule number 2 is called "the associative property of addition and multiplication".
Rule number 3 is called "the distributive property of multiplication over addition "
(and also known as the "factoring rule".)
Rule number 4 is called "the existence of an inverse in addition "
Rule number 5 is called "the existence of an inverse in multiplication "

In the Rational Numbers every number except zero has an inverse under multiplication. The inverse of 3 is ⅓ because 3 x ⅓ = 1, and 1 is the identity of rational numbers under multiplication. Dividing by 3 is the same as multiplying by the inverse of 3, which is ⅓.

12) Complete these statements about arithmetic with matrices:
i) Addition and multiplication of matrices is __________.

ii) Addition of matrices is _________ but multiplication of matrices is not.

iii) Every number and every matrix has an __________ under addition.

Let's go back and look at our equation $\begin{pmatrix} 2+1 \\ 1-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 8 \\ 1 \end{pmatrix}$. You solved this by multiplying it out into two equations and solving them simultaneously. We are going to solve this by matrix multiplication. The special matrix $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has it's own name - the "Identity" matrix.

13 i) Calculate $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

ii) What do IA equal and AI equal, where A is any matrix?

14) The two matrices $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ go together in a special way.

i) Show that $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}=I$ and $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}=I$.

ii) Show that $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

iii) Show that if $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$ then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

(Note that $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ may be more neatly written as $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$.)

iv) Hence find the solution x and y to equation 1.

Matrix $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ is called the "*inverse*" of $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$. If a matrix A has an inverse then there is a matrix B such that AB and BA both equal I. The inverse of A, if it exists, is written A^{-1}.

E 15) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

i) Show that the equation AB = I gives four equations, and the first two are $ae + bg = 1$ and $af + bh = 0$.

ii) By writing $f = -\frac{bh}{a}$ and substituting, show that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

iii) Check that the product of these two matrices, when multiplying in either order, is I.

E 16) What problems can arise with this formula for the inverse? In other words, in what conditions will it not be valid? What does this tell you about the "inverse" of the matrix in this case?

In the formula for the inverse of the 2 x 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the expression "$ad - bc$" is so important that it has its own name. It is call the "determinant" of the matrix. The determinant of matrix A is written $\det(A)$.

17) Complete these sentences:

i) Every number has an __________ under __________ but not every matrix has an ________________ under ____________.

ii) If the determinant of a matrix is ________ the matrix will not _______________.

iii) If the determinant of a matrix is not ________ the matrix will _______________.

Note that if a matrix has no inverse we say it is "singular". If it has an inverse it is called "nonsingular".

18 i) Show that the determinant of the matrix $\begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix}$ is not zero.

ii) Conclude from (i) that the matrix $\begin{pmatrix} 3 & 1 \\ 0 & -5 \end{pmatrix}$ has an inverse and find the inverse.

ii) Check that the product of the matrix and its inverse, both ways round, equals the Identity matrix.

19 i) Try and find the solution, if it exists, to this set of simultaneous equations:

$3x + 2y = 1$

$6x + 4y = 3$ What is the " answer" here?

ii) Show that the equation can be written in matrix form as $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

iii) Does the matrix $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ have an inverse?

iv) Show that the same matrix could be used to find the solution, if it exists, to this set of simultaneous equations:

$3x + 2y = 12$

$6x + 4y = 24$ What is the " answer" here?

20) Complete these sentences:

i) If the matrix of a system of simultaneous equations has an ________ then the equations will have a u_______ s_______________.

ii) If the matrix of a system of simultaneous equations does not have an ________ then the equations will have either n_______ s_______________s or m_________ s_________s.

Note that a matrix with no inverse is called a "singular" matrix.

21) This is a simultaneous equation in three variables

$x + 3y + z = 8$

$2y + 2z = -4$

$x - z = 5$

i) Show that it can be converted to the matrix equation $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

ii) Show that the solution will be given by the matrix equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

iii) Use your calculator in matrix mode to show that $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \frac{3}{2} & 2 \\ 1 & -1 & -1 \\ -1 & \frac{3}{2} & 1 \end{pmatrix}$.

iv) Hence find the solution to the equation.

Completing the solution to the equation above requires us to find the inverse of a 3 x 3 matrix, which you can do with your calculator. But you need to know the (rather complicated) formula for finding the inverse of a 3 x 3 matrix by hand. Before you can do this you first need to know how to calculate the *determinant* of a 3 x 3 matrix.

22) Follow the method given below to calculate the determinant of the matrix $M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}$.

i) The determinant of a 3 x 3 matrix depends on repeated calculation of the determinant of a 2 x 2 matrix. Return to question 15 to ensure that you know the formula for the determinant of a 2 x 2 matrix.

ii) Draw a line through the *top row* of matrix M and the through the *first* column of matrix M . The lines will intersect at M_{11}. (By M_{11} we mean the value in row 1, column 1 of the matrix. In this $M_{11} = 1$.) Calculate the determinant of the remaining terms, that is: $\det\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}$.

iii) Draw a line through the *top* row of matrix M and the through the *second* column of matrix M . The lines will intersect at M_{12}. (By M_{12} we mean the value in row 1, column 2 of the matrix. In this $M_{12} = 3$.) Calculate the determinant of the remaining terms, that is: $\det\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

iv) Draw a line through the *top* row of matrix M and the through the *third* column of matrix M . The lines will intersect at M_{13}. Calculate the determinant of the remaining terms, that is: $\det\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

iv) Calculate the determinant from the formula $\det M = M_{11}\det\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} - M_{12}\det\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} + M_{13}\det\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

v) Check your answer with a calculator.

23) By applying the same formula to the matrix $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, show that

$$\det M = a\times\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\times\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\times\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}.$$

The reduced matrices, obtained by deleting a row and a column are called "cofactors".
C_{11} is the symbol for the cofactor obtained by deleting row 1 and column 1.
C_{ij} is the symbol for the cofactor obtained by deleting row i and column j.

24) Show that the formula above may be written as $\det(M) = a\times\det(C_{11}) - b\times\det(C_{12}) + c\times\det(C_{13})$

In fact, we could have got the same value for the determinant by, instead of deleting the *first* row to obtain each cofactor, deleting the *second* row or the *third* row. However, we need a change of signs when we do that. Notice that the signs go "+ - +" in the formula above when we multiply each cofactor. If we delete the second row the signs go "- + -". The pattern of signs across the matrix goes $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. This pattern will also be important when we come to calculate the inverse of a 3 x 3 matrix.

25 i) We can call the matrix of "+" and "-" signs above "S" the "Sign matrix". Show that S_{ij} = "+" if $(i+j)$ is even and "-" if $(i+j)$ is odd.

ii) Show that the statement in (i) is equivalent to the formula $S_{ii} = (-1)^{i+j}$

26) Show that the formula for the determinant may also be written as

$$\det(M) = -d\times\det(C_{21}) + e\times\det(C_{22}) - f\times\det(C_{13}) \text{ and}$$

also as $\det(M) = g\times\det(C_{31}) - h\times\det(C_{32}) + i\times\det(C_{33})$

27) Show that the general formula for the determinant of a 3 x 3 matrix M is:

$$\det(M) = M_{k1}S_{k1}C(M)_{k1} + M_{k2}S_{k2}C(M)_{k2} + M_{k3}S_{k3}C(M)_{k3}$$

where S is the sign matrix, C(M) is the cofactor matrix of M and k can be either 1 or 2 or 3.

As you can see, there is quite a bit of work to calculate a *determinant* for a 3 x 3 matrix. (Your calculator will do it for you. Use it to check.) But it gets worse. Now we must calculate the *inverse* of a 3 x 3 matrix. Recall that the “cofactors” of a matrix are the submatrices obtained by deleting a row and a column and making a matrix of what's left. So cofactor C_{ij} of matrix the 3 x 3 matrix M is the 2 x 2 matrix obtained by deleting row i and column j from M.

28) $M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix}$

i) Show that M has nine cofactors.
ii) Write down cofactors C_{13} and C_{22} and their determinants.
iii) The “cofactor matrix” of M, which we will call C(M), is the matrix obtained by replacing each element of M by its respective cofactor. So that $\left(C(M)\right)_{ii} = \det\left(C_{ii}\right)$.
Write the matrix C(M)..

The “transpose” of a matrix is the matrix made by swapping the rows and columns of the matrix, so that row 1 becomes column 1, row 2 becomes column 2 and so on. We write the transpose of a matrix A as A^T.

29 i) Show that the transpose of matrix M above is $M^T = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & -1 \end{pmatrix}$.

ii) Write down the transpose of the cofactor matrix $\left(C(M)\right)^T$.

Finally we are able to specify the form of the inverse of M. The inverse of matrix M is the *transpose of the cofactor matrix of M multipied by the reciprocal of the determinant of M.*

In other words $M^{-1} = \frac{1}{\det(M)}\left(C(M)\right)^T$.

30 i) Write down M^{-1} , - the inverse of the matrix M.
ii) Confirm that $MM^{-1} = I$ and $M^{-1}M = I$, where I is the identity matrix.

31 i) Matrix $N = \begin{pmatrix} k & 3 & 1 \\ 1 & k & 2 \\ 2 & 1 & -1 \end{pmatrix}$, where k is a real number. Calculate det(N) in terms of k.

ii) State the values of k for which N is singular.
iii) Assuming that N is non-singular, calculate N^{-1} in terms of k.
iv) If N^{-1} has all the same values on the main diagonal (that is at position 1,1, 2,2 and 3,3 in the matrix) find the possible values of k.