Chapter 4: Polynomials and their roots

1) Sketch the graph of the curve with these equations :

i) $y = x^2 - 5x + 6$

ii) $y = x^2 - 6x + 9$

iii) $y = x^2 - 6x + 12$

In each case mark the vertex of the curve and all intercepts with the axes.

2) Check your answers with a graphing application, if necessary.

The equations above are called "quadratic equations", the name coming from the Latin word "quadratus" meaning a square. The curve of a quadratic equation is called a "parabola". The points where the curve crosses the x- axis are called the "real roots" of the equation.

3) Explain why in question 1 equation i) has two real roots, equation ii) has one (duplicated) real root and equation iii) has no real roots.

4) The general quadratic equation may be written as $y = ax^2 + bx + c$ where a, b and c are real numbers, and a is not zero.

i) Show that if $ax^2 + bx + c = 0$ then $x = \frac{1}{2a}\left(b \pm \sqrt{D}\right)$ where $\frac{b}{2a}$ equals the x coordinate of the vertex of the quadratic curve and $D = b^2 - 4ac$ is called the "discriminant" of the equation.

ii) Relate the value of D to your answer to question 2 (if you have not already done so.)

iii) A "complex conjugate pair" are two complex numbers of the form a + bi and a - bi, where b is not zero. Based on the value of the discriminant of the quadratic, conclude that the two roots of a quadratic will be either two distinct real roots, one double real root or a complex conjugate pair.

5 i) Show that $ax^2 + bx + c$ may be written in the form $\left(x - \left(\frac{1}{2a}\left(b + \sqrt{D}\right)\right)\right)\left(x - \left(\frac{1}{2a}\left(b - \sqrt{D}\right)\right)\right)$.

ii) Write the three equations in question 1) in the form $\left(x - \left(\frac{1}{2a}\left(b + \sqrt{D}\right)\right)\right)\left(x - \left(\frac{1}{2a}\left(b - \sqrt{D}\right)\right)\right)$.

6 i) Write the equation $ax^2 + bx + c = 0$ in the form $x^2 + px + q = 0$, stating the values of p and q.

ii) Write the roots of the equation $x^2 + px + q = 0$ in terms of p and q.

iii) Show that the *sum of the roots* of $x^2 + px + q = 0$ *equals p* and the *product of the roots* of $x^2 + px + q = 0$ *equals q*.

7) The roots of a polynomial with integer coefficients are α *and* $\frac{2}{\alpha}$. Find α and write the polynomial.

8) Sketch the graph of the curve with these equations :

i) $y = -3x^3$

ii) $y = (x-1)(x+2)(x+4)$

iii) $y = (x-1)(x^2 + 3x + 4)$

In each case mark all the intercepts with the axes and state the number of "real roots" of each equation.
Check your answers with a graphing application.

The equations above are called "cubic equations". Both cubic and quadratic equations are examples of "polynomials". A polynomial is a series of powers of x multiplied by constants known as "coefficients". In this case the coefficients will always be real numbers (not complex numbers.) The "degree" of the polynomial is the highest power of x. So a quadratic is a polynomial of degree 2. A cubic is a polynomial of degree 3. The next ones are called "quartics" and "quintics", after the Latin names for four and five.

9 i) Expand the brackets in the cubic equation $y=(x-1)(x+2)(x+4)$ and write it in the form $f(x)=x^3+px^2+qx+r$.
ii) The roots of $f(x)$ above are α, β and χ where $\alpha=1, \beta=-2, \chi=-4$. Show that in the expansion of $f(x)$, $\alpha+\beta+\chi=-p$ and $\alpha\beta\chi=-r$ and $\alpha\beta+\alpha\chi+\beta\chi=q$.
iii) Expand the brackets in the cubic equation $y=(x-1)(x+2)(x+4)$ and write it in the form $f(x)=x^3+px^2+qx+r$. Show that $\alpha+\beta+\chi=-p$ and $\alpha\beta\chi=-r$ and $\alpha\beta+\alpha\chi+\beta\chi=q$, where α, β and χ are the roots of $f(x)$.

10) You will have learnt the "Factor Theorem" which says that if $f(x)$ is a polynomial and $f(a)=0$ for some value a, then $(x-a)$ is a factor of $f(x)$.
i) If $f(x)=x^3+x^2-2$ show that $f(1)=0$, hence factorise $f(x)$ into $(x-1)g(x)$ where $g(x)$ is a quadratic.
ii) Since a quadratic has either real roots or complex conjugate roots, show that $f(x)$ has either three real roots or one real root and a complex conjugate pair.
iii) Write the three roots of $f(x)$ and factorise $f(x)$ fully.

11 i) Using $f(x)$ inQuestion 10, calculate the *sum* of the three roots of $f(x)$ and the *product* of the three roots of $f(x)$.
ii) Compare the sum and product of the roots with the *coefficients* of $f(x)$ and note whatever you notice. Return to question 6 iii) and relate what you notice to this.

The general cubic is a polynomial of the form ax^3+bx^2+cx+d, where a, b, c and d are real numbers but *x* could be complex. Finding the "roots" of the cubic is equivalent to solving the equation $ax^3+bx^2+cx+d=0$. It is usually easier to do this by dividing through by "a" first and so change the equation to $x^3+px^2+qx+r=0$. The Factor Theorem tells us that if the roots of $f(x)$ are α, β and χ then $f(x)$ may be written as $(x-\alpha)(x-\beta)(x-\chi)$.

12 i) Show that the equation $ax^3+bx^2+cx+d=0$ may be written as $x^3+px^2+qx+r=0$, stating the values of p, q and r.
ii) If α, β and χ are the roots of $f(x)=x^3+px^2+qx+r$ write $f(x)=(x-\alpha)(x-\beta)(x-\chi)$ and expand out the brackets to show that $\alpha+\beta+\chi=-p$ and $\alpha\beta\chi=-r$ and $\alpha\beta+\alpha\chi+\beta\chi=q$ ..

13 The cubic $f(x)$ has roots $1, (3-i), (3+i)$. Write $f(x)$ as $x^3+px^2+qx+r=0$.

You will have noted that the complex roots above are $(3+i)$ and $(3-i)$. And you will also have noted that if the roots of a quadratic are complex then they are $(a+bi)$ and $(a-bi)$, for some real numbers a and b. In other words, they come in pairs and these *pairs* are called *complex conjugate pairs.* In fact, complex roots of *any* polynomial *always* come in complex conjugate pairs. This is an important point. You won't be required to prove this but it will help you're learning if you understand why it happens.

(E) We will try and demonstrate that if $(a+bi)$ is a root then so is $(a-bi)$. In other words, if $f(a+bi)=0$ then $f(a-bi)=0$.

E 14 i) Expand $(3+i)^3$ with the Binomial Theorem. Show that it equals $(u+vi)$.

ii) Expand $(3-i)^3$ with the Binomial Theorem. Show that it equals $(u-vi)$.

E 15 i) Expand $(a+bi)^3$ with the Binomial Theorem. Expand $(a-bi)^3$ with the Binomial Theorem.

ii) Compare the terms of each expansion. When are they the same? When are they different and how are they different?

iii) Write $(a+bi)^3$ as $(u+vi)$, where u and v are in terms of a and b. Write $(a-bi)^3$ in the same way and show that it equals $(u-vi)$.

iv) What if you tried $(a+bi)^2$ and $(a-bi)^2$? Would you find the same thing? What if you tried $(a+bi)^4$ and $(a-bi)^4$? Would you find the same thing?

Note that this is only true if the coefficients of $f(x)$ *are real. We consider only polynomials with real coefficients.*

v) Given that $f(a+bi)=u+vi$, confirm the validity of this statement:

"*If* $(a+bi)$ *is a root of* $f(x)$ *then* $f(a+bi)=0$, *which means that* $(u+vi)=0$, *which means that* $u=0$ *and* $v=0$. $f(a-bi)=u-vi$, *but* $u=0$ *and* $v=0$ *because* $(a+bi)$ *is a root, so* $f(a-bi)=0$. *So* $(a-bi)$ *is a root.*"

E 16) Conclude from the above statement that in any cubic the three roots will either be all real or one will be real and the others a complex conjugate pair.

17 i) $f(x)$ is the polynomial $x^3-3x-52$. $(-2+3i)$ is one root of $f(x)$. Find the other two.

ii) Two roots of a polynomial $f(x)=x^3+px^2+qx+r$ are 3 and $(2-i)$. Write the polynomial.

18 i) Solve the equation $x^3+1=0$. (You can see one root and work out the others.)

ii) Mark the roots on an Argand diagram.

It should be fairly clear by now that you can completely solve a cubic if you are given one root or can spot one. You can often make a guess at the first root if it is a small integer, such as x = -1 in question 18. If there is not a simple root then finding one could be quite tricky. (Your calculator should have an Equation mode that will solve any cubic for you, but use that just for checking.)

In a quadratic you have a formula (sing it to the tune of "Happy Birthday") which will always give you both roots of the quadratic equation. In fact there is a formula for the cubic but it's quite complicated. I have the greatest respect for the mathematicians of the sixteenth century who worked it out when algebra was in its infancy and there were no recognised symbols or formats for equations that are so familiar to us today. Look up the names of Scipione del Ferro, Niccolò Tartaglia and Girolamo Cardano, Franciscus Vieta and Muhammed Al Khwarizmi. You will find an explanation of the method as it was discovered over time, and there is a good story behind it.

19 i) The polynomial $ax^4+bx^3+cx^2+dx+e$ is called a "quartic". The equation $ax^4+bx^3+cx^2+dx+e=0$ may be written as $x^4+px^3+qx^2+rx+s=0$, writing the coefficients p, q, r and s.

ii) If the roots of the quartic are α, β, χ and δ the quartic may be written as $(x-\alpha)(x-\beta)(x-\chi)(x-\delta)$. Show that
$\alpha+\beta+\chi+\delta=-p, \quad \alpha\beta+\alpha\chi+\alpha\delta+\beta\chi+\beta\delta+\chi\delta=q, \quad \alpha\beta\chi+\alpha\beta\delta+\alpha\chi\delta+\beta\chi\delta=-r, \quad \alpha\beta\chi\delta=s$.

iii) Noting that complex roots always come in pairs, show that $\alpha, \beta, \chi, \delta$ will be all real roots or else two pairs of complex conjugates.

20) Two roots of a quartic are $(2-i)$ and $(-3+2i)$. Write down the quartic.

21 i) Return to your sketch of $y=(x-1)(x+2)(x+4)$ in question 8. Another cubic has roots at $\frac{1}{2}, -1$ and -2, half the roots of this one. Sketch the curve of this cubic.

ii) If the curve of the first function with roots at 1 , -2 and -4 was $y=f(x)$, show that the curve of the second function is $y=g(x)=f(2x)$. (You will have learnt the rules of transformation of functions in A-Level mathematics.) Write $g(x)$ as x^3+px^2+qx+r, giving the values of p, q and r.

iii) A third function, $h(x)$, has roots at x_h = 5, -1 and -5. There is a linear relationship between the roots of $h(x)$ and the roots of $f(x)$ (x_f = 1, -2, -4) such that $x_f=ax_h+b$. Find a and b.

iv) Noting that $h(x)$ may be written as $k(x-5)(x+1)(x+5)$ for some value k, show that $h(x)$ may be written as $h(x)=f\left(\frac{x-a}{b}\right)$ for those values a and b.

v) Write $h(x)$ as x^3+px^2+qx+r, giving the values of p, q and r.