Chapter 5: Number Series

This chapter covers two chapters in Further Mathematics FP1 Chapter 5 on Number Series and FP2 Chapter 4 on summing series with thee Method of Differences.

1 i) What is 1 + 2 + 3 + 4?

ii) What is 1 + 2 + ... + 10?

iii) What is 1 + 2 + ... + *n*, as a formula involving *n*? Explain your answer.

iv) What is 1 + 3 + 5 + ... + 19?

v) What is $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$, as a formula involving *a*, *d* and *n*? Explain your answer.

All of the series in question 1 are "arithmetic series", which means there is a fixed difference between successive terms. You should know the formula for the nth term and sum of n terms of an arithmetic series, but it is in the formula book.

We have a simple way of writing the series 1 + 2 + ... + 10. We write it as $\sum_{k=1}^{10} k$. Note that you have a very useful button on your calculator marked with that same sign. Use it to check your work in question 1.

2 i) Show that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

ii) Show that series 1 + 3 + 5 + ... + 21 in question 1 can be written as $\sum_{k=1}^{10}(2k-1)$ and sum the expression using the result in part (i).

3 i) Write the expression $a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$, in the form $\sum_{k=1}^{n} f(k)$, stating $f(k)$ in terms of a and d.

ii) Use the result in question 2 to show that $\sum_{k=1}^{n} f(k)$ equals $\frac{1}{2}n(2a+(n-1)d)$.

4) Work out $\sum_{k=1}^{10}(3k+2)$ by two different methods.

i) The first way is to write out $\sum_{k=1}^{10}(3k+2)\cdots(3\times1+2)+(3\times2+2)+\cdots+(3\times10+2)$

Show that this sum equals 5 + 8 + ... + 32 and sum it as a normal arithmetic series.

ii) The second way is to recognise that $\sum_{k=1}^{10}(3k+2)=\sum_{k=1}^{10}3k+\sum_{k=1}^{10}2$. Show that this sum equals the sum in part (i).

5 i) What is $1^2+2^2+3^2+4^2$?

ii) What is $1^2+2^2+\cdots+10^2$? (Hint: There is a button on your calculator to save time.)

6) Consider the series $S(n)=(2^2-1^2)+(3^2-2^2)+\cdots+((n+1)^2-n^2)$.

i) Show that $(k+1)^2-k^2=2k+1$ and so show that $S(n)=\sum_{k=1}^{n}(2k+1)$.

ii) Show that the series may also be written as:

$$\begin{aligned}&2^2-1^2\\&+3^2-2^2\\&+4^2-3^2\\&+\cdots\cdots\\&+n^2-(n-1)^2\\&+(n+1)^2-n^2\end{aligned}$$

and that if you cancel plus terms and minus terms you get $S(n)=(n+1)^2-1=n^2+2n$.

iii) Noting from (i) that $S(n)=\sum_{k=1}^{n}(2k+1)=2\sum_{k=1}^{n}k+n$ and from (ii) that $S(n)=n^2+2n$,

show that $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$, which of course you already knew.

I am sure you know a simpler method than this one to prove that $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$. But we have used this one here to introduce the "method of differences". It is used for proving the sums of a series and it relies on writing the series as a sum of differences which cancel, as in the example above. We are going to derive a formula for $\sum_{k=1}^{n}k^2$ using the same method.

7) Consider the series $S(n)=(2^3-1^3)+(3^3-2^3)+\cdots+((n+1)^3-n^3)$

i) Show that $(k+1)^3-k^3=3k^2+3k+1$ so that

$$S(n)=\sum_{k=1}^{n}(3k^2+3k+1)=3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+n=3\sum_{k=1}^{n}k^2+\frac{n}{2}(3n+5)$$

ii) Show that $S(n)$ may also be written as

$$\begin{aligned}&2^3-1^3\\&+3^3-2^3\\&+4^3-3^3\\&+\cdots\cdots\\&+(n-1)^3-(n-2)^3\\&+n^3-(n-1)^3\end{aligned}$$

And that if you cancel plus terms and minus terms you get $S(n)=(n+1)^3-1=n^3+3n^2+3n$

iii) Noting from (i) that $S(n)=3\sum_{k=1}^{n}k^2+\frac{n}{2}(3n+5)$,

and from (ii) that $S(n)=(n+1)^3-1=n^3+3n^2+3n$,

conclude that $3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+n=n^3+3n^2+3n$, and hence that $\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)$.

8) Derive the formula for $\sum_{k=1}^{n}k^3$ using the same method as above, and if you feel like it, do it again to find the formula for $\sum_{k=1}^{n}k^4$.

9) Work out $\sum_{k=1}^{n}\left(3k^2+2k+1\right)$ by splitting it into three separate series - a quadratic term $\left(n^2\right)$, a linear term (n) and a constant term (1) and summing each one.

10) By using the cubic series formula work out:

i) $\sum_{k=1}^{6}k^3$ ii) $\sum_{k=1}^{6}\left(3k^3-1\right)$ iii) $\sum_{k=7}^{15}\left(3k^3-1\right)$

11) Consider the sequence $S(n)=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots\cdots+\frac{1}{n}-\frac{1}{n+1}$.

i) By grouping the sequence like this: $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$,

show that $S(n)=\sum_{k=1}^{n}\frac{1}{n(n+1)}$.

ii) By grouping the sequence like this $\frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots\cdots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}$ show that $S(n)=1-\frac{1}{n+1}$ and so conclude that $\sum_{k=1}^{n}\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.

12) Now consider any function $f(n)$ and another function $g(n)=f(n)-f(n+1)$.

Show that $\sum_{k=1}^{n}g(k)=f(1)-f(n+1)$.

13 i) Show that $\frac{1}{k!}-\frac{1}{(k+1)!}=\frac{k}{(k+1)!}$.

ii) Hence show that $\sum_{k=1}^{n}\frac{k}{(k+1)!}=\sum_{k=1}^{n}\frac{1}{k!}-\sum_{k=1}^{n}\frac{1}{(k+1)!}$.

iii) Show that $\sum_{k=1}^{5}\frac{k}{(k+1)!}=\frac{119}{120}$ and that $\sum_{k=6}^{10}\frac{k}{(k+1)!}=0.0083333$.

14 i) Show that $\frac{1}{x^2+3x+6}=\frac{1}{x+2}+\frac{1}{x+3}$.

ii) Hence show that $\sum_{k=1}^{n}\frac{1}{x^2+3x+6}=\frac{1}{3}-\frac{1}{n+4}$.