Chapter 6: Taylor Series and Maclaurin Series

In Chapter 4 we looked at *polynomials* - a series of powers of x up to the "nth degree". But what if the polynomial does not terminate so that powers of x are unbounded?

1) Consider this series:- $f(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots$

i) What is the next term in the series?

ii) By putting enough terms into your calculator calculate $f(1)=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$

Does the answer to this question look like a number that is familiar to you?

iii) Write down the derivative $f'(x)$. Did the result surprise you?

iv) Do you know another function that is equal to its own derivative? (There is actually only one.)

Since there is only one function that is equal to its own derivative and that is "Euler's function" $f(x)=e^x$ we should conclude that the series above is e^x.

2) Enter the series into your calculator and by using the CALC button or the Table mode convince yourself that $f(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots$ does give the same values as e^x.
(You could also use a spreadsheet for this.)

3) Consider these two functions:
$c(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots$ and $s(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots$

i) Write down the next term in each series.

ii) Calculate $c(\pi)$ and $s(\pi)$ and $c\left(\frac{\pi}{2}\right)$ and $s\left(\frac{\pi}{2}\right)$ approximately, using the first few terms of each series.

iii) Calculate the derivatives $c'(x)$ and $s'(x)$.

iv) Write $c'(x)$ in terms of $s(x)$ and $s'(x)$ in terms of $c(x)$.

I am sure you have figured out by now that $c(x)$ and $s(x)$ are the functions $\cos(x)$ and $\sin(x)$ in series form. But how did anyone discover these series? They were discovered in the early eighteenth century and the method was published in a book by the English mathematician Brook Taylor, entitled in Latin "*Methodus incrementorum directa et inversa*" in 1715. But they received most attention when they were taken up and used widely by the Scottish mathematician Colin Maclaurin in his book "Treatise of Fluxions" in 1742.

4) Suppose we have a function $f(x)$ and we want to write this function as an infinite series
$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$

i) Show that $f(0)=a_0$, $f'(0)=a_1$, $f''(0)=2a_2$, etc

ii) Calculate $f^n(0)$ in terms of n and the coefficients a_n.

iii) Rearrange the formulas above to state each coefficient $a_0, a_1, \cdots,$ in terms of n and $f^n(0)$.

iv) Substitute your expressions for $a_0, a_1, \cdots,$ into the series $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots$

In question 4 you should have the formula $f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\cdots$

This is the Maclaurin Series for x.

5) We want to find the coefficients $a_0 , a_1 , \cdots$, so that the function e^x can be written as the Maclaurin series $e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

i) Show that $f(0) = a_0 = 1$, $f'(0) = a_1$, $f''(0) = 2a_2$, etc.

ii) Noting that $f(x) = f'(x) = f''(x) = f'''(x)$ etc , show that $f(0) = f'(0) = f''(0) = f'''(0) = \cdots\cdots = 1$ f(0)

iii) Show that $a_n = \dfrac{1}{n!}$, n = 0, 1, 2, ... which means that $e^x = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \cdots$

6) We want to find the coefficients $a_0 , a_1 , \cdots$ so that the trigonometric functions can be written in the Maclaurin series $\cos(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ and $\sin(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

i) Show that the values of $\cos(0)$ and the first, second and third derivatives of $\cos(x)$ at $x = 0$ are, respectively, 1, 0, -1 and 0 and that these values equal a_o , a_1 , $2a_2$ and $6a_3$ respectively.

ii) Hence write down the infinite series for $\cos(x)$.

iii) Show that the values of $\sin(0)$ and the first, second and third derivatives of $\sin(x)$ at $x = 0$ are, respectively, 0, 1, 0 and -1 and that these values equal b_o , b_1 , $2b_2$ and $6b_3$ respectively.

iv) Hence write down the infinite series for sin(x).

7 i) Check the series by showing that $\cos'(x) = -\sin(x)$ and $\sin'(x) = \cos(x)$ and that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. ($\cos(x)$ is an *even* function, $\sin(x)$ is an *odd* function.)

ii) By putting enough terms into your calculator confirm some key values of $\sin(x)$ and $\cos(x)$.

8) $f(x) = \dfrac{1}{1-x}$

i) Show that $f'(x) = \dfrac{1}{(1-x)^2}$, $f''(x) = \dfrac{2!}{(1-x)^3}$ and $f^n(x) = \dfrac{n!}{(1-x)^{n+1}}$, n = 1, 2, 3, ...

ii) Hence show that $f^n(0) = n!$, n = 1, 2, 3, ...

iii) Hence write down the infinite series for $\dfrac{1}{1-x}$.

iv) In what context have you seen this function and its infinite series before?

9) Find the Maclaurin series for $f(x) = \dfrac{1}{1+x}$.

10) i) Write the Maclaurin series for $\ln(1+x)$.

ii) Confirm from the series for $\dfrac{1}{1+x}$ that the derivative of $\ln(1+x)$ is $\dfrac{1}{1+x}$.

11 i) By substituting x^2 for x in the series for $\ln(1+x)$, write the Taylor series for $\ln(1+x^2)$.

ii) By substituting x^2 for x in the series for $\dfrac{1}{1+x}$, write the Taylor series for $\dfrac{1}{1+x^2}$.

iii) Confirm from the series that the derivative of $\ln(1+x^2)$ is $\dfrac{2x}{1+x^2}$.

11 i) $\ln(1+x^2) = 1 - x^2 + \dfrac{1}{2}x^4 - \dfrac{1}{3}x^6 + \cdots$ $\qquad f(x) = f(0) + f'(0)x + \dfrac{1}{2!}f''(0)x^2 + \dfrac{1}{3!}f'''(0)x^3 + \cdots$

12 i) Find the first four terms of the series for $e^{\sin(x)}$.

(Hint: $e^{\sin(x)} = e^{a_0+a_1x+a_2x^2}$ where $a_0 + a_1x + a_2x^2$ are the first three terms of the expansion of $\sin(x)$.

ii) Calculate an estimate of $e^{\sin\left(\frac{1}{2}\right)}$ from the series. Then compare this with your calculator value and calculate the percentage error.

We found series for $\ln(1+x)$ and $\ln(1-x)$. It raises the question why we did not start with the simpler function $\ln(x)$. What problem are you going to meet if you try and use the formula to find the series expansion of $\ln(x)$?

This series is centred on zero and so requires calculation of $f(x)$ at $x=0$. A *Maclaurin Series* is a *Taylor Series centred on zero*. But a Taylor Series can be centred on any point x for which $f(x)$ is defined. This is useful when the function is not defined at $x = 0$, making the Maclaurin series unworkable. But it can also be useful at other times to expand a series about a particular point.

13) We will now find the the Taylor Series expansion of $\ln(x)$.

Make the substitute variable $u = 1+x$, or $x = u-1$. Then $\ln(u) = \ln(1+x)$. Substitute $(u-1)$ for x in the expansion of $\ln(1+x)$ to get the expansion of $\ln(u)$. Then, since the letter used for the variable is arbitrary, replace u by x and get the Taylor Series expansion for $\ln(x)$, about $x = 1$.

14) Expand the function $\frac{1}{x}$ in a Taylor Series about $x = 1$. Show from the Taylor Series for $\ln(x)$ that the derivative of $\ln(x)$ is $\frac{1}{x}$.

15 i) Expand the functions $\sin(x)$ and $\cos(x)$ in a Taylor Series about $x = \frac{\pi}{2}$.

ii) Confirm that the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$.

iii) Show that $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$ and $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$.