Chapter 7: Further Complex Numbers - Euler's Identity and de Moivre's Theorem

1 i) $z = (1 - 3i)$. Find the modulus (r) and argument (θ) of z.

ii) Write z in the form $r\,\text{cis}(\theta) = r(\cos(\theta) + i\sin(\theta))$.

iii) Display z on an Argand diagram.

In Chapter 6 you learnt how to write functions in the form of an infinite series. In particular, you wrote the functions e^x, $\cos(x)$ and $\sin(x)$ in the form of an infinite series (called a Maclaurin series.)

2 i) Write e^{ix} as a Maclaurin series. (Be careful with powers of "i".)

ii) Re-arrange your series into the form (Real Part) + i(Imaginary Part).

iii) Show that "Real Part" is the Maclaurin series for $\cos(x)$ and "Imaginary Part" is the Maclaurin series for $\sin(x)$.

iv) Conclude that $\text{cis}(x) = \cos(x) + i\sin(x) = e^{ix}$.

The statement above is "Euler's Identity", first discovered by Leonhard Euler and published in 1748 in Euler's book "Introduction to Analysis of the Infinite". This uses the definition of a complex number to link functions which nobody thought up till then would have a connection - the exponential function e^x and the trigonometric functions $\sin(x)$ and $\cos(x)$. The physicist Richard Feynman called this formula "our jewel" and "the most important formula in mathematics."

We now know four different ways of writing a complex number :

$z = (a + bi)$, $z = r(\cos(x) + i\sin(x))$, $z = r\,\text{cis}(x)$ and $z = re^{ix}$.

3 i) Write $z = (1 - 3i)$ in the form $z = re^{i\theta}, -\pi < \theta < \pi$.

ii) Show that z could also be written as $z = re^{i(\theta + 2\pi)}, -\pi < \theta < \pi$ or as $z = re^{i(\theta - 2\pi)}, -\pi < \theta < \pi$.

$-\pi < \theta < \pi$ is called the "principal range" of the argument of z. The argument of a complex number is defined to be in this range. So it is always $-\pi < \theta < \pi$ unless stated otherwise.

4) Write $z = re^{i\frac{3\pi}{4}}$ in the form $r(\cos(\theta) + i\sin(\theta))$ and then in the form $(a + bi)$.

5) Assuming that the index rules for complex powers work just the same as the index rules for real powers (which they do) show that if $z_1 = re^{i\theta}$ and $z_2 = re^{i\varphi}$ then $z_1 z_2 = re^{i(\theta+\varphi)}$ and $\frac{z_1}{z_2} = re^{i(\theta-\varphi)}$,

and hence $(z_1)^n = re^{in\theta}$ and $\frac{1}{(z_1)^n} = re^{-in\theta}$.

6) If $z=(a+bi)$ then the complex conjugate of z is $z^*=(a-bi)$.

i) Show that if $z=r\,\text{cis}\,\theta$ then $z^*=r\,\text{cis}(-\theta)$.

(Hint: You could make use of the fact that cos(x) is an even function and sin(x) is an odd function.)

ii) Show that if $z=re^{i\theta}$ then $z^*=re^{-i\theta}$.

iii) Noting that $|z|=zz^*$, show that $|z|=r^2$.

7) i) Take $z=(1-3i)$ and calculate $2z, z^2, z^{-1}$. Then write $z=re^{i\theta}$ and do the same calculations with the same answers but in "Euler form".

ii) Take $z_1=(1-3i)$ and $z_2=(3+4i)$ and calculate z_1+z_2, z_1z_2 and $\dfrac{z_1}{z_2}$.

iii) Your calculator should be able to do complex arithmetic and swap between a + bi mode and Euler mode. Use your calculator to check your answers.

8) If $z=re^{i\theta}$ show that $z^n=r^ne^{in\theta}=z^n=r^n\text{cis}(n\theta)$.

Question 8 gives us the statement $\left(r\left(\cos(\theta)+i\sin(\theta)\right)\right)^n=r^n\left(\cos(n\theta)+i\sin(n\theta)\right)$ which is a very important theorem called De Moivre's Theorem. Abraham de Moivre was a French mathematician who, because life was difficult for a Protestant in France at the time, lived most of his life in London. He was known for his insights into the new field of Probability, as well as his famous theorem which he published in 1722. In terms of Euler's identity, de Moivre's theorem seems obvious - simply a matter of applying the index rules. But de Moivre's theorem predated Euler. We will prove de Moivre's Theorem by induction.

9 i) Look up a geometric proof of the well-known identities $\cos(a+b)=\cos^2 a-\cos^2 b$ and $\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b)$.

ii) Use these identities to show that $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$ and $\sin(2\theta)=2\sin(\theta)\cos(\theta)$

iii) de Moivre's theorem is clearly true when n = 1.

Assume that $\left(r\left(\cos(\theta)+i\sin(\theta)\right)\right)^n\equiv r^n\left(\left(\cos(n\theta)+i\sin(n\theta)\right)\right)$ for n = 1, 2, 3, ...

Now show that $\left(r\left(\cos(\theta)+i\sin(\theta)\right)\right)^{n+1}\equiv r^{n+1}\left(\left(\cos((n+1)\theta)+i\sin((n+1)\theta)\right)\right)$ and complete the proof by induction.

10 i) Write $z=(1-3i)$ in the form $z=r\,\text{cis}\,\theta$.

ii) Show that z may also be written as $r\,\text{cis}(\theta+2\pi)$, $r\,\text{cis}(\theta+4\pi)$ and $r\,\text{cis}(\theta+6\pi)$.

iii) Use de Moivre's Theorem to solve the equation $z^4 = 1 - 3i$, giving all four solutions in the form $s\,\text{cis}(\varphi)$, where $s = r^{\frac{1}{4}}$.

11 i) If $z = \text{cis}(\theta)$ express each of $z + \frac{1}{z}, z - \frac{1}{z}, z^n + \frac{1}{z^n}, z^n - \frac{1}{z^n}$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

ii) Show that $\sin^4(\theta) = \frac{1}{16}\left(z - \frac{1}{z}\right)^4$.

iii) Apply the binomial expansion to show that $\sin^4(\theta) = \frac{1}{16}\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$.

iv) Use de Moivre's theorem to show that $\sin^4(\theta) = \frac{1}{8}\left(\cos(4\theta) - 2\cos(2\theta) + 3\right)$

v) Find the integral of $\sin^4(\theta)$.

12 i) If $z = \cos(\theta) + i\sin(\theta)$ then $z^4 = \cos(\theta) + i\sin(\theta)^4$. Expand this with a binomial expansion into the form $z^4 = a(\theta) + b(\theta)i$, where $a(\theta)$ and $b(\theta)$ are functions of θ.

ii) By de Moivre's Theorem $z^4 = \cos(4\theta) + i\sin(4\theta)$. So $\cos(4\theta) = a(\theta)$ and $\sin(4\theta) = b(\theta)$. Hence express $\cos(4\theta)$ and $\sin(4\theta)$ as a series of powers of $\sin(\theta)$ and $\cos(\theta)$.

13 i) Show that $\text{cis}(2n\pi)$, n = 1, 2, ... 6 all equal 1.

ii) Use de Moivre's theorem to solve $z^6 = 1$. In other words, find the six roots of unity.

iii) Mark the six roots on an Argand diagram and describe the geometric shape the roots make.

14) Find the six roots of i and mark them on an Argand diagram.

We are going to make a connection between sets of points on the complex plane, as defined by certain rules, and familiar geometrical shapes. This is what is generally known as "locus problems".

15 i) Show that the "modulus" of a complex number, $|z|$, is its distance from the origin.

ii) Show that the distance between two complex numbers z_1 and z_2 is $|z_1 - z_2|$.

16) Sketch on x-y axes the locus of points at a distance 2 units from the origin. What is the Cartesian equation of this curve? Now sketch the same curve on Real and Imaginary axes. What statement about a complex number is true for all complex numbers that lie on the curve and false for all complex numbers that do not lie on the curve? Express this statement as an equation in z.

17) Sketch on x-y axes the locus of points at a distance of 2 units from the point (2,1). What is the equation of this curve? Now sketch the same curve be an Argand diagram on Real and Imaginary axes. It is the locus of points at a distance of 2 units from the point $(2+i)$. What statement about a complex number z is true for all complex numbers that lie on the curve and is false for all complex numbers that do not lie on the curve? Express this statement as an equation in z and the number $(2+i)$.

18 i) Consider a vertical line drawn drawn on x-y axes through the point $x = 2$. That, of course, is the equation of the line. Consider the same line drawn on Real and Imaginary axes. Express this statement as an equation in z. (You can use $\mathrm{Re}(z)$ to mean the Real part of z and $\mathrm{Im}(z)$ to mean the Imaginary part of z.)

ii) Consider a horizontal line drawn drawn on Real and Imaginary axes through $y = -1$. Write the equation of the line as an equation in z .

19 i) Draw Real and Imaginary axes and mark two points anywhere on the axes. Call these points z_1 and z_2 . Mark a point that is *equidistant* from both points. Now mark another point that is equidistant from both points, and another….. What geometric shape will you draw if you keep going?

ii) If z is a complex number express in an equation the statement "The distance of z from z_1 is equal to the distance of z from z_2".

iii) This equation in question (ii) is the equation of a straight line. State the geometric relationship between this line and the points of z_1 and z_2.

20) i) Draw on an Argand diagram the locus of points given by $|z-6|=|z|$.

ii) Draw on an Argand diagram the locus of points given by $|z-3i|=|z|$.

iii) Draw on an Argand diagram the locus of points given by $|z-3i|=|z-6|$.

iv) Draw on an Argand diagram the locus of points equidistant from the points $4i$ and -2. Give the equation of the line in terms z and also the Cartesian equation of the line.

21) Using your diagrams above solve the two simultaneous equations $|z-3i|=|z-6|$ and $|z-4i|=|z+2|$ and so find the intersection of the two lines.

22) Consider the locus of points given by $|z-6i|=2|z-3|$.

i) By writing z as $(x+yi)$ and using the formula for the modulus of z, write this as an equation in x and y.

ii) Show that it is an equation of a circle and give the centre of the circle and its radius.

iii) Using your diagrams above solve the two simultaneous equations $|z-4i|=|z+2|$ and $|z-6i|=2|z-3|$.

23) i) Consider a line that begins at the origin and extends straight out at an angle of $\frac{\pi}{4}$ to the x-axis. This line is called a "half -line" or "ray". What statement about a complex number z is true for all complex numbers that lie on the line and is false for all complex numbers that do not lie on the line? Express this statement as an equation in z. Hint: Think *argument* of z $(\arg(z))$.

ii) Draw the locus of points given by the equation $\arg(z)=\frac{3\pi}{4}$.

iii) Solve the two simultaneous equations $|z-4i|=|z+2|$ and $\arg(z)=\frac{3\pi}{4}$.