Chapter 8: Matrices and Transformations

1) ABCD is a rectangle with co-ordinates at A(0,0), B(0,2), C(1,2), D(1,0).
i) Draw this rectangle on the x-y axes.
ii) Create a 2 x 4 matrix, R, whose columns are the coordinates of each corner of the rectangle. So the first column is (0, 0), the second is (0,2) and so on.
iii) Let T be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Calculate the matrix R' = TR.
iv) Draw the shape A'B'C'D' where A'B'C'D' are the columns of matrix R'.

This new shape is a transformation of the rectangle called a *shear*. We call the original rectangle the *object*. The "sheared" rectangle is the it's *image* under the transformation T. T is called the *transformation matrix.*

2 i) Let V be the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Calculate the matrix R'' = VR.
ii) Draw the shape A''B''C''D'' where A''B''C''D'' are the columns of matrix R''.
iii) Describe the transformation from R to R''.
(Note that almost all the things we will do in this chapter can be done quite quickly on a graphics app such as Geogebra. You can define a shape with coordinates and create a matrix to multiply those coordinates into a new shape. I encourage you to use an app like Geogebra to check your working.)

3 i) Calculate the inverse of matrix V and apply the inverse of V to matrix R''.
ii) Describe the transformation from R'' to R.

4 i) Let VT be matrix V times matrix T. Write matrix VT.
ii) Calculate the matrix R' = (VT)R, where R is the matrix in question 1.
iii) Draw the shape A'B'C'D' where A'B'C'D' are the columns of matrix R'.
iv) Describe the two transformations that took R to R'.

5 i) Calculate the inverse of matrix VT and apply it to matrix R' in question 4.
ii) What shape do you get?
iii) Show that $(\mathrm{VT})^{-1} = \mathrm{T}^{-1}\mathrm{V}^{-1}$.

6 i) Let TV be matrix T times matrix V. Write matrix TV.
ii) Calculate the matrix R'' = (TV)R, where R is the matrix in question 1.
iii) Draw the shape A''B''C''D'' where A''B''C''D'' are the columns of matrix R''.

7 i) Calculate the inverse of matrix VT and apply it to matrix R'' in question 6.
ii) What shape do you get?
iii) Show that $(\mathrm{TV})^{-1} = \mathrm{V}^{-1}\mathrm{T}^{-1}$.

The transformations we have seen are all "linear transformations". Linear transformations preserve straight lines as straight lines (though not usually the same straight line.) The linear transformations are ones that transform a shape in familiar ways such as rotations, reflections, enlargements and shears.

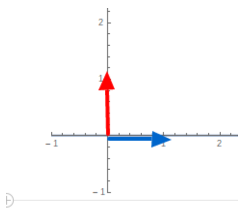
We have seen that a shape can be represented by a matrix of points and a linear transformation of the shape can be represented by a square matrix. The "inverse" transformation is represented by the inverse matrix. Two transformations in succession can be represented by the product of the two matrices.

How can we find the exact matrix that will perform a particular transformation? This turns out to be easier than you might think.

A good way to determine the effect of a transformation is to take two very simple shapes - a *unit vector* in the direction of the *positive y-axis* and a *unit vector* in the direction of the *positive x-axis* - and look at the effect of the transformation on those.

8) The unit vector in the direction of the positive x-axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The unit vector in the direction of the positive y-axis is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We will call these two vectors the “Identity vectors”.

i) Draw the base vectors on x-y axes, as shown here: .



I have distinguished the horizontal vector from the vertical vector with colours. When drawing these you should use some method for distinguishing one from the other. For example, you could give the horizontal vector two arrows and the vertical vector one arrow.

ii) Write the matrix formed by the base vectors as columns.

You will see that the matrix of Identity vectors is the Identity matrix, which is very convenient because it tell us immediately the effect of any transformation on the Identity vectors.

9) V is the transformation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

i) Show that the Image matrix of the Identity vectors after transformation V is the matrix V itself.
ii) Draw the image of the Identity vectors and describe the transformation.

10) $R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

i) Draw the image of the Identity vectors under transformation by each of R_1, R_2 and R_3.
ii) Describe each transformation.

11) By noting the effect of the transformation on the Identity vectors, describe the transformation represented by each of these matrices:

i) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ ii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ iii) $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

12 i) Draw the Identity vectors on an x-y grid.
ii) Show the Image of the Identity vectors under the transformation “Rotation 45° anti-clockwise about the origin”.
iii) Write the Image vectors as column vectors and hence write the matrix representing a rotation 45° degrees anti-clockwise about the origin.

13) Use the same method as question 12 to write the matrix which represents the following transformations :
i) Reflection in the y-axis and enlargement scale factor 2.
ii) Rotation 180° and enlargement scale factor 1/2, centre the origin.

14 i) Apply the transformations R_1, R_2 and R_3 to the original rectangle in question 1.
ii) Confirm that the transformation is as described in question 10.
iii) Show that in each case the area of the image is the same as the area of the object.
iv) Matrices R_1, R_2 and R_3 in question 10 each have a determinant equal to 1. Confirm this.

15 i) Apply the transformations $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ from question 11 to the original rectangle in question 1.

ii) Confirm that the transformation is as described in question 11.
iii) In each case calculate the *ratio* of the area of the image to the area of the original rectangle. (This ratio is called the *scale factor for area* of the transformation.)
iv) Calculate the determinant of each transformation matrix above.
v) Show that the scale factor for area is equal to the magnitude of the determinant. (The magnitude is the positive value, whether the given value is positive or negative.)
vi) Calculate the determinant of matrices $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ and $R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For what transformation do these matrices apply? Does the sign of the determinant tell you anything about the type of transformation?
vii) The *scale factor* of a transformation is the scale factor of *lengths* not of *areas*. What is the scale factor of the transformations $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$?
vii) Make a connection between the *scale factor* of the transformation and the determinant of the transformation matrix.
viii) Show that the transformations represented by matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ do not change area.