Chapter 9: Integration for volumes, lengths of curves and surface area

1 i) Sketch the graph of the curve $y = e^x$ across the full range of x.

ii) Calculate these integrals $\int_{-1}^{0} e^x dx, \quad \int_{-5}^{0} e^x dx, \quad \int_{-10}^{0} e^x dx$.

iii) By noting the answers to the integrals make a guess at the value of $\int_{x=-\infty}^{0} e^x dx$.

$\int_{x=-\infty}^{0} e^x dx$ is an example of an "improper integral". It cannot be computed directly because ∞ is not a number. But it does have an exact value that can be calculated as the *limit* of a function. Calculating limits of functions is the key to the subject of Analysis which you will study more of at University. We will consider limits in a "common sense" manner without worrying too much about precise details.

2 i) Write $\int_{x=-\infty}^{0} e^x dx$ as $\int_{x=-M}^{0} e^x dx$ and show that this equals $1-e^{-M}$.

ii) Convince yourself by whatever means necessary that as M gets large e^{-M} will approach zero. That means that $1-e^{-M}$ approaches 1.

3) Calculate $\int_{x=-\infty}^{0} 2^x dx$ (Hint: $a^x = e^{x\ln(a)}$)

4) Calculate $\int_{x=1}^{\infty} \frac{1}{x^2} dx$ and $\int_{x=1}^{\infty} \frac{1}{x^3} dx$.

Do not think this is all as simple as it looks. I have not asked you to calculate $\int_{x=1}^{\infty} \frac{1}{x} dx$ yet.

5 i) Calculate $\int_{x=1}^{10} \frac{1}{x} dx, \quad \int_{x=1}^{20} \frac{1}{x} dx$ and $\int_{x=1}^{50} \frac{1}{x} dx$. Show that the integral appears to increase without limit.

ii) Show that $\int_{x=1}^{M} \frac{1}{x} dx = \ln(M)$ and since $\ln(x)$ is an unbounded function (which means it increases without limit) then $\int_{x=1}^{\infty} \frac{1}{x} dx$ is an unbounded integral .

6 i) By considering the values of $\int_{x=M}^{1} \frac{1}{x} dx$ for values of M near zero show that $\int_{x=0}^{1} \frac{1}{x} dx$ is unbounded.

ii) Show that $\int_{x=0}^{1} \frac{1}{x^n} dx$ is unbounded if $n \geq 1$.

7 i) Sketch the curve of $y = \dfrac{1}{\sqrt{x}}$ from $x = 0$ to 1. Shade the area above the curve and below the x-axis.

ii) Show that $\int_{x=0}^{1} \frac{1}{\sqrt{x}} dx = 2$.

8 i) Let's return to a simpler, more familiar integral.

Sketch the curve $y=(x-1)(5-x)$ and shade the area below the curve and above the x-axis.
ii) Draw a rectangle between the points x = 1 and x = 5. Set the height of the rectangle so that the area of the rectangle is the same as the area under the curve.

If you calculated the last question correctly you must have calculated that the area of the rectangle was 4h, where h is the height, and so if the two areas are to be equal then $h=\frac{1}{4}\int_{x=1}^{5}(x-1)(5-x)dx$. What you have done here is to calculate the "mean value of the function". You will be familiar with mean values from Statistics. Suppose $f(x)$ was a *discrete* function calculated on on the interval [1,5], not a continuous one. You would calculate the mean value of $f(x)$ by summing up $f(x)$ for all the values x and dividing by the number of points.

9 i) Make a table showing x for values 1, 2, 3, 4, 5 and $f(x)$ evaluated for each point.
ii) Calculate $\frac{1}{5}\sum_{x=1}^{5}f(x)$. How close are you to your mean value, h, in question 8?
The answer is $\frac{1}{5}\sum_{x=1}^{5}f(x)=\frac{10}{5}=2$, which is not very close to $\mathrm{h}=\frac{8}{3}$. But try it again with 10 values of x in the interval [1,5] in steps of $\frac{1}{2}$. You could use the table function on your calculator, or a spreadsheet. You will find you get closer.

10) Now imagine a function, $f(x)$, defined for a set of discrete points $x_0, x_1, \cdots x_n$ on the interval [a, b] so that $x_0=\mathrm{a}$ and $x_n=\mathrm{b}$.
i) Show that if the points are a fixed distance, δx, apart then $n=\frac{b-a}{\delta x}$. (Can you see why I started counting the points at x_0 not x_1 ?)
ii) Show that the mean of the values is = $\frac{1}{n}\sum_{k=0}^{n}f(x_k)=\frac{\delta x}{b-a}\sum_{k=0}^{n}f(x_k)=\frac{1}{b-a}\sum_{k=0}^{n}f(x_k)\delta x$.
iii) Show that the limit of this series as n approaches infinity and δx approaches zero is the integral of $f(x)$ on the interval [a, b] and so the formula for the *Mean Value of f(x)* must be $\frac{1}{b-a}\int_{x=a}^{b}f(x)\delta x$.

If you continue to study mathematics you will find that the mean value of a function will crop up in various contexts. In particular, there is a *Mean Value Theorem for Integrals* which states that "*If f(x) is continuous on an interval [a, b] then there will be at least one point, x, within that interval where f(x) equals its mean value on the interval.*"

There is also a *Mean Value Theorem for Derivatives* which would be worth your while looking up.

11) Return to your sketch of a curve and a rectangle in question 7 and mark the point where $f(x)$ equals its mean value. Conclude that the mean value theorem is intuitively obvious.

We are now going to look at methods that use the integral to find volumes of solid shapes, surface areas of solid shapes and lengths of curves. This extends the methods you already know for finding area with integrals. You can calculate all these quantities - area, volume, surface area and length quite quickly with a graphing app. In the questions that follow I invite you to check your answers with a graphing app, like Desmos. This will also help you to draw your own diagrams better and to visualise the geometry more clearly.

12 i) Sketch the curve of $y=x^2, 0\le x\le 1$.

ii) Now sketch a "trumpet shape" formed by rotating that curve 2π radians round the x-axis. (This is a "solid of revolution about the x-axis".)
iii) Show in your diagram how you could divide your solid "trumpet shape" up into a number of thin disks, standing vertically on their edge with their centre on the x-axis.
iv) Call the width of each thin disk δx and show that the radius of each disk is equal to $y = x^2$. Hence show that the volume of each disk is $\pi y^2 \delta x$.

v) Since the trumpet shape is a sum of many of these thin disks write the volume as $\sum_{k=1}^{n} \pi y^2 \delta x$ and that as n gets large δx will get small and, in the limit, this sum will equal $\int_0^1 \pi y^2 dx$.

vi) Noting that $y = x^2$, show that the volume of the trumpet shape is $\int_0^1 \pi x^4 dx$ and calculate the volume.

The solid shape you have drawn is known as a "solid of revolution about the x-*axis*" because you have rotated it around the x-axis. Any continuous curve above the x-axis can be revolved like this and if the curve is $y = f(x)$ defined between the values of $\mathrm{a} \le x \le \mathrm{b}$ then the volume of the associated solid of revolution will have a volume equal to $\int_a^b \pi y^2 dx$.

13 i) Sketch the curve of $y = x^2, 0 \le x \le 1$.
ii) Now sketch a "bowl shape" formed by rotating that curve 2π radians round the y-axis. (This is a "solid of revolution about the y-axis".)
iii) Show in your diagram how you could divide your solid "bowl shape" up into a number of thin horizontal disks, stacked in a pile like plates with their centre on the y-axis.
iv) Call the height of each thin disk δy and show that the radius of each disk is equal to $x = \sqrt{y}$. Hence show that the volume of each disk is $\pi x^2 \delta y = \pi y \delta y$.

v) Show that the volume of the bowl shape is $\int_{y=0}^{1} \pi y \, dy$ where $y = x^2$ and calculate the volume.

The solid shape you have drawn is known as a "solid of revolution about the y-axis" because you have rotated it around the y-axis. Any continuous curve can be revolved like this and if the curve is $y = f(x)$ defined between the values of $\mathrm{a} \le x \le \mathrm{b}$ then the volume of the associated solid of revolution will have a volume equal to $\int_a^b \pi |y| \, dy$. (The modulus of y is taken because areas below the x-axis are negative in integration.)

14 i) Sketch the curve $y = \sqrt{1 - x^2}$ on $0 \le x \le 1$. Show that this curve is a quarter circle.
ii) Calculate the volume of a bowl shape formed by rotating the curve 2π radians about the y-axis.

iii) Show that the volume of the bowl is $\int_0^1 \pi (1 - y^2) dy$ and that this equals $\frac{2}{3}\pi$.

iv) How could you know this answer before you did the calculation?

15 i) Sketch the curve in parametric form, $x(t) = \frac{1}{4}t^3, \quad y(t) = t, \quad -2 \le t \le 2$.
ii) Sketch the solid shape formed by rotating the curve 2π radians about the x-axis.

iii) Noting that $dx = \frac{dx}{dt} dt$, show that the volume of the shape is given by $\int_{x=-2}^{2} \pi \left(\frac{1}{4}t^3\right)^2 \left(\frac{3}{4}t^2\right) dt$.

iv) Simplify this expression and calculate the volume.

I remember when I was first learning about functions, curves and integration that I was puzzled as to why we were being taught to find the *area* under a curve or the *volume* created by a curve before we were taught what seemed the more obvious thing - the *length* of a curve. The reason for this is that the calculation of area arises naturally as the integral of a function. For the sake of completeness, and because it is interesting (at least I think so) this is demonstrated below, but it is not part of the syllabus and you will not be asked to reproduce this in an exam.

E 15) If $f(u)$ is a function of variable u then the area under the curve of $y = f(u)$ for $0 \le u \le x$ is a well-defined function of x. Sketch a graph of $y = f(u)$ for some function (eg for $f(u) = u^2, 0 \le u \le x$) and shade the area under the curve between 0 and x. This area is a function of x, so call it $\mathrm{A}(x)$.
ii) Show that this area can be approximated by a sum of small rectangles of area with a base of width δu and a height of $y = f(u)$. Hence the area of the rectangle is $f(u)\delta u$.
iii) Since the shaded area is a sum of many of these thin rectangles, show that you can write the area as $\sum_{k=1}^{n} f(u)\delta u$ and that as n gets large then δx will get small and, in the limit, this sum will equal $\int_0^x f(u)\delta u$ which is $\mathrm{A}(x)$.

We are now going to use integration to find *lengths* of curves. But the integral does not arise so naturally and directly from the function $f(x)$ as it does with area. Often the integral cannot be evaluated exactly and must be calculated numerically.

16) i) Sketch the curve of $y = x^2, 0 \le x \le 1$. (Draw it a good size so that you can zoom in on a small part of the curve.)
ii) Mark two points on the curve, P, where x = a and Q, where x = b. We want to measure the length of that piece of curve between P and Q. (We call it an “arc”.) If the arc is small you can approximate it with a straight line and calculate the length of that line with Pythagoras. Draw a *chord* between points P and Q. (A chord is a straight line joining points on a curve.) Make a right angle triangle with your chord as the hypotenuse. Label the horizontal side of the triangle δx and the vertical side δy.
iii) Show that the length of the chord you have drawn is $\sqrt{(\delta x)^2 + (\delta y)^2}$ and that we could measure the length of the curve between 0 and 1 as the sum of many such small chords.
iv) Show that the length of the arc between P and Q is approximated by the sum of many such small chords and so may be written as $\sum_{k=1}^{n} \sqrt{(\delta x)^2 + (\delta y)^2}$.
v)The more chords you make, the smaller they will be and the closer your estimate of the length will come to the true value. So that as n gets large then δx and δy will get small and, in the limit, this sum will equal $\int_{x=a}^{b} \sqrt{(dx)^2 + (dy)^2}$. Show that the length of the curve between a and b is

$$\int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{(dx)^2 \left(1 + \left(\tfrac{dy}{dx}\right)^2\right)} = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx .$$

17 i) Sketch the curve $y = x^2$ and highlight the arc of the curve from $x = 0$ to $x = 1$. Using the formula $\mathrm{L} = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx$ for the length of the arc between a and b, show that the length of the arc of $y = x^2$ from $x = 0$ to $x = 1$ is given by the integral $\mathrm{L} = \int_0^1 \sqrt{1 + 4x^2}\, dx$.

ii) You will learn how to compute this integral in the later chapter on hyperbolic functions. For now, compute it numerically with your calculator.

18 i) Sketch the curve in parametric form $x(t)=\frac{1}{4}t^3,\ y(t)=t, -2\le t\le 2$.

ii) Noting that $dx=\dfrac{dx}{dt}dt$ and that $dy=\dfrac{dy}{dt}dt$, show that the formula for the length of a curve written in parametric form is $\mathrm{L}=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.

iii) Hence show that the length of this curve is given by $\mathrm{L}=\int_{-2}^{2}\sqrt{\left(\frac{3}{4}t^2\right)^2+1}\,dt$.

iv) Calculate this numerically with your calculator..

We are now going to use the same technique as above to calculate the *surface area* of the trumpet shape and the bowl shape.

19 i) Sketch the curve of $y=x^2, 0\le x\le 1$.

ii) Now sketch a "trumpet shape" formed by rotating that curve 2π radians round the x-axis .

iii) Show in your diagram how you could divide the *surface area* of your "trumpet shape" up into a number of thin disks, each standing vertically on their edge with their centre on the x-axis .

iv) The surface area of each thin disk is equal to its circumference multiplied by its width. Show that the circumference is equal to $2\pi y$ and the width is equal to a length of the small part of the curve, which we can call δs , and hence the surface area of the thin disk is $2\pi y\,\delta s$.

iv) Since δs is a small length of the curve, show by analogy with the formula in question 17, that $\delta s=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

v) Since the surface area of the shape is the sum of small disks of area $2\pi y\,\delta s$, show that the surface area of the shape is $\mathrm{A}=\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, where $y=x^2$.

vi) Substitute the values for y and $\dfrac{dy}{dx}$ to the formula above and calculate the integral.

20) We have shown that the surface area of a 3-dimensional shape created by rotating a curve, $y=f(x), \mathrm{a}\le x\le \mathrm{b}$ around the x-axis has the formula $\mathrm{A}=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

i) If the curve is given in parametric form as $x=x(t),\ y=y(t),\ t_1\le t\le t_2$ show that the surface area has the formula $\mathrm{A}=\int_{t_1}^{t_2} 2\pi\,|\,y(t)\,|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$. The modulus of y is taken because $y(t)$ may be negative.

21 i) Sketch the curve in parametric form, $x(t)=\frac{1}{4}t^3, y(t)=t, -2\le t\le 2$.

ii) Sketch the 3D shape formed by rotating the curve 2π radians about the x-axis.

iii) Show that the surface area of this shape is $\mathrm{A}=\int_{-2}^{2} 2\pi\,|\,t\,|\sqrt{\left(\frac{3}{4}t^2\right)^2+1}\,dt=2\int_0^2 2\pi t\sqrt{\left(\frac{3}{4}t^2\right)^2+1}\,dt$.

iv) By making the substitution $u=\frac{3}{4}t^2$, show that this integral equals $\frac{8}{3}\pi\int_0^3\sqrt{1+u^2}\,du$ and evaluate the integral.