90.68 Yet another look at the calculus power rules

In [1], Sidney Schuman asks, for the particular case $y = x^n$, the question: '*Why* is an area rule the inverse of a gradient rule?' The real reason for this, even for more general functions, is the way that the area under a graph is intimately linked to the distance of each point on the curve from the x-axis, and the way that this is in turn linked to the gradient of the curve at each point. Let us show what we mean by this with the help of Figure 1 below, which is an extended and generalised version of Figure 1 in [1]. Note that the curve divides the rectangle formed by lines perpendicular to the coordinate axes from P into regions A and B, and a right triangle with base c is formed by the dashed line tangent to the curve at P. Although Schuman is only interested in the case $f(x) = x^n$, let us, for the moment at least, consider the slightly more general situation whereby $f(x)$ is a non-decreasing, differentiable function of x for $x \geqslant 0$, with $f(0) = 0$.

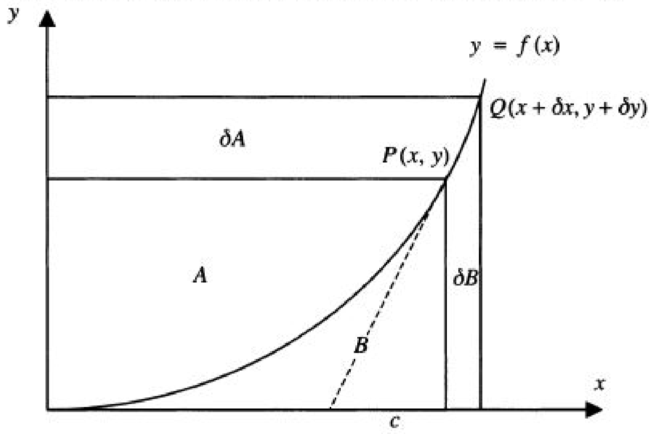

FIGURE 1

We have that $y\delta x < \delta B < (y + \delta y)\delta x$, giving us

$$y < \frac{\delta B}{\delta x} < y + \delta y.$$

On letting $\delta x \to 0$ we obtain $dB/dx = y$ and, as a consequence,

$$\frac{d^2B}{dx^2} = \frac{d}{dx}\left(\frac{dB}{dx}\right) = \frac{dy}{dx},$$

showing immediately why an area rule is the inverse of a gradient rule. Of course, this inverse relationship is particularly obvious when $f(x) = x^n$, since in this case we have the simple result $f'(x) = nx^{n-1}$.

Admittedly, however, the above explanation is not entirely within the spirit of [1]. There the author notes that, for the case $f(x) = x^n$, the relation $A/B = n$ is obtained from the integral power rule while the relation $x/c = n$ is obtained from the differential power rule (refer to Figure 1). He seeks an answer to his question via a mathematical formulation, in terms of 'dynamic' geometry, of the connection between these relations, making the comment

that as P moves along the curve the tangent line 'projects' the ratio A/B onto the x-axis.

Adopting this 'dynamic' point of view then, we may envisage the tangent at $P(x, y)$ sliding up the curve, with areas A and B gradually increasing as a consequence. Say that the tangent has slid from $P(x, y)$ to $Q(x + \delta x, y + \delta y)$, and that δA and δB are the respective increases in the areas of A and B. Let us firstly assume that $A/B = n$ for all $x > 0$. Then, since $A/B = n$ and $(A + \delta A)/(B + \delta B) = n$, we obtain $\delta A/\delta B = n$. However, we also have that $x\delta y < \delta A < (x + \delta x)\delta y$ and $y\delta x < \delta B < (y + \delta y)\delta x$, giving us

$$\frac{x\,\delta y}{(y + \delta y)\,\delta x} < n < \frac{(x + \delta x)\,\delta y}{y\,\delta x}.$$

On letting $\delta x \to 0$ we obtain $dy/dx = ny/x$. Next, from the definition of c, we have that $dy/dx = y/c$. Equating these two expressions for dy/dx gives $x/c = n$.

If, on the other hand, we assume that $x/c = n$ then, using $dy/dx = y/c$ once more, we have $dy/dx = ny/x > 0$. Again $x\delta y < \delta A < (x + \delta x)\delta y$ and $y\delta x < \delta B < (y + \delta y)\delta x$, so that

$$\frac{x\delta y}{(y + \delta y)\,\delta x} < \frac{\delta A}{\delta B} < \frac{(x + \delta x)\,\delta y}{y\delta x}.$$

Now, letting $\delta x \to 0$, we obtain

$$\frac{dA}{dB} = \frac{x}{y}\frac{dy}{dx} = \frac{x}{y} \times \frac{ny}{x} = n,$$

which implies, since $f(0) = 0$, that $A/B = n$.

Reference

1. Sidney Schuman, Another look at the calculus power rules, *Math. Gaz.* **89** (July 2005) p. 251.

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