Differential equations

If the relationship between two variables is not known, but the derivative of one with respect to the other *is* known, the solution of the given differential equation can be used to find the relationship. Only first-order differential equations (which do not include second derivatives) are covered on this course and these come in three types:

Type I: [dy/dx $= f(x)$] solved by direct integration.

Type II: [dy/dx $= f(x)g(y)$] solved by integration of both sides after separation of variables.

Type III: [dy/dx = k*y*] - solved by integration of both sides after separation of variable and constant.

Solutions of differential equations may be in **explicit** form stated as $y = f(x)$, or in **implicit** form where the function is implied but not stated. An arbitrary constant of integration is present in a **general** solution, but not in a **particular** solution where given initial conditions have been substituted to evaluate the constant. In the examples below the codes IGS, EGS, IPS and EPS have been used to indicate combinations of implicit, explicit, general and particular solutions.

Type I

Find the particular solution of $\frac{dy}{dx} = \frac{1}{\sqrt{x}} - \frac{1}{x}$ if $y = 3$ when $x = 1$. $y = \int \left(x^{-\frac{1}{2}} - \frac{1}{x} \right) dx = 2\sqrt{x} - \ln x + C$ (EGS)

Substitute initial conditions: $3 = 2\sqrt{1} - \ln(1) + C \quad \therefore C = 1 \qquad \therefore y = 2\sqrt{x} - \ln x + 1$ (EPS)

Type II

(a) Find the particular solution of $\frac{1}{2y}\frac{dy}{dx} = -3x^2$ if $y = 4$ when $x = 0$.

$\frac{1}{2}\int \frac{1}{y} dy = \int -3x^2 dx \quad \therefore \frac{1}{2}\ln y = -x^3 + C$ (IGS) $\quad \therefore \ln y = -2x^3 + C \quad \therefore y = e^{-2x^3 + C}$ (EGS)

$\therefore y = Ae^{-2x^3}$ (where $A = e^c$) Substitute initial conditions: $4 = Ae^{-2(0)^3} \quad \therefore A = 4 \quad \therefore y = 4e^{-2x^3}$ (EPS)

(b) Find the particular solution of $y \sin x \frac{dy}{dx} = \cos x$ if $y = 1$ when $x = \frac{\pi}{4}$.

$\int y\, dy = \int \cos x \sin x\, dx \quad \therefore \frac{1}{2}y^2 = \frac{1}{2}\cos(2x) + C \quad \therefore y^2 = C - \cos(2x)$ (IGS) $\quad \therefore y = \sqrt{C - \cos(2x)}$ (EGS)

Substitute initial conditions: $1 = \sqrt{C - \cos(\frac{\pi}{2})} \therefore C = 1 \qquad \therefore y = \sqrt{1 - \cos(2x)}$ (EPS)

Type III

Find the particular solution of $\frac{dy}{dx} = -5y$, given that $y = 2$ when $x = 0$

$\int \frac{1}{y} dy = \int -5 dx \quad \therefore \ln y = -5x + C$ (IGS) $\quad \therefore y = e^{-5x+c} = e^{-5x} \times e^c \quad \therefore y = Ae^{-5x}$ (EGS where $A = e^c$)

Substitute initial conditions: $2 = Ae^0 \therefore A = 2 \qquad \therefore y = 2e^{-5x}$ (EPS)

Exercises - find explicit particular solutions for the following differential equations:

1) $\frac{dy}{dx} = 8x - 6\cos(2x)$ if $y = -5$ when $x = 0$

2) $\frac{dy}{dx} = e^{\frac{1}{2}x} + 2\cos x$ if $y = 2$ when $x = 0$

3) $\frac{dy}{dx} = \frac{x}{y^2}$ if $y = 3$ when $x = 4$

4) $\frac{dy}{dx} = xy^2$ if $y = 1$ when $x = 0$

5) $\frac{dy}{dx} = 0.05y$ if $y = 500$ when $x = 0$

6) $\frac{dy}{dx} = e^{-2y}$ if $y = 0.5$ when $x = 0$.

Answers: $y =$

1) $4x^2 - 3\sin(2x) - 5$ **2)** $2e^{\frac{1}{2}x} + 2\sin x$ **3)** $\sqrt[3]{\frac{3}{2}x^2 + 3}$ **4)** $\frac{2}{2 - x^2}$ **5)** $500e^{0.05x}$ **6)** $\ln\sqrt{2x + e}$