Integration by parts

Integration of products of functions, except where one function is the derivative of the other, must be done using a technique that is based on the integration of both sides of the product rule of differentiation. This product rule is: $\left(f(x)g(x)\right)' = f(x)g'(x) + f'(x)g(x)$ and integrating it gives: $f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx$

Re-arranging this gives the formula for integration by parts: $\underline{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$.

Type I $f'(x)$ is a constant (or combines with $g(x)$) and the last part can be integrated directly.

Type II $f'(x)$ is a function different to $f(x)$ and integration by parts must be used more than once.

Type III $f(x)$ is an exponential function, $g'(x)$ is a trigonometric function, integrate twice and combine.

Examples

Type I $\int 3x\cos(5x)dx = \frac{3}{5}x\sin(5x) - \int \frac{3}{5}\sin(5x)dx = \frac{3}{5}x\sin(5x) + \frac{3}{25}\cos(5x) + C = \underline{\frac{3}{25}\left(5x\sin(5x) + \cos(5x)\right) + C}$

{ Here we chose $f(x) = 3x$ and $g'(x) = \cos(5x)$ giving us $f'(x) = 3$ and $g(x) = \frac{1}{5}\sin(5x)$ }

$\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C = \frac{1}{16}x^4\left(4\ln x - 1\right) + C = \underline{\frac{1}{16}x^4\left(\ln x^4 - 1\right) + C}$

{ Here we chose $f(x) = \ln x$ and $g'(x) = x^3$ giving us $f'(x) = \frac{1}{x}$ and $g(x) = \frac{1}{4}x^4$ }

Type II $\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x\cos x\, dx$ $\quad \int 2x\cos x\, dx = 2x\sin x - \int 2\sin x\, dx = 2x\sin x + 2\cos x + C$

{Here we chose (1st): $f(x) = x^2$ and $g'(x) = \sin x$, then (2nd): $f(x) = 2x$ and $g'(x) = \cos x$ }

$\therefore \int x^2 \sin x\, dx = -x^2 \cos x + 2x\sin x + 2\cos x + C = \underline{2x\sin x + \cos x(2 - x^2) + C}$

Type III $\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$ {with $f(x) = e^x$ and $g'(x) = \cos x$ } , then:

$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx \therefore \int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$, so adding these we get: $2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C$ $\quad \therefore \int e^x \cos x\, dx = \underline{\frac{1}{2}e^x(\sin x + \cos x) + C}$

{By a similar process, $\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$ }

Definite integration by parts

Limits must be attached to all parts as shown here: $\int_b^a f(x)g'(x)\, dx = \left[f(x)g(x)\right]_b^a - \int_b^a f'(x)g(x)dx$

Example 1 $\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 e^x dx = \left[xe^x\right]_0^1 - \left[e^x\right]_0^1 = \left(1\times e^1 - 0\times e^0\right) - \left(e^1 - e^0\right) = e - e + 1 = \underline{\mathbf{1}}$

Example 2 $\int_0^{\frac{\pi}{2}} x\sin x\, dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \left[\sin x\right]_0^{\frac{\pi}{2}} = \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right) + \left(\sin\frac{\pi}{2}\right) = \underline{\mathbf{1}}$

Example 3 $\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \frac{1}{2}\left[\left(e^x(\sin x - \cos x)\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\left(e^{\frac{\pi}{2}}(\sin\frac{\pi}{2} - \cos\frac{\pi}{2})\right) - \left(e^0(\sin 0 - \cos 0)\right)\right) = \frac{1}{2}\left(e^{\frac{\pi}{2}} + 1\right) = \underline{\mathbf{2.905}}$

Exercises

1) $\int_0^1 xe^{2x}dx$ 2) $\int_{\frac{\pi}{2}}^{\pi} (1+x)\sin(3x)dx$ 3) $\int_0^1 xe^{4x}dx$ 4) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\sin(2x)dx$

5) $\int_1^2 x^2 \ln(4x)dx$ 6) $\int_0^1 x^2e^{5x}dx$ 7) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} x\cos(6x)dx$ 8) $\int_0^{\frac{\pi}{4}} e^{\frac{1}{2}x}\cos(2x)dx$

Answers: 1) 2.097 2) 1.492 3) 10.3 4) 0.5 5) 4.305 6) 20.168 7) 0 8) 0.579