Linear Second-Order Recurrence Sequences

This worksheet is about finding the closed form of a recurrence system of the type:
$u_{n+2} = p \times u_{n+1} + q \times u_n$ (n = 0, 1, 2, ...), the values of the first two terms u_0, u_1 being given.

How to do it

1 Write down an auxiliary quadratic equation of the form: $x^2 - px - q = 0$
Solve this equation to find the roots, α and β
If $\alpha \neq \beta$ *(distinct roots) the type of closed form will be:* $u_n = A\alpha^n + B\beta^n$
If $\alpha = \beta$ *(repeated roots) the type of closed form will be:* $u_n = (A + Bn)\alpha^n$

2 Form a pair of simultaneous equations using the given values of the first two terms and solve them to evaluate the constants A and B

Example 1
Find the closed form for the recurrence system: $u_0 = 2, u_1 = 7, u_{n+2} = 2u_{n+1} + 8u_n \quad (n = 0, 1, 2, ...)$ and use it to find the value of the 8th term.
Auxiliary equation: $x^2 - 2x - 8 = 0 \quad \therefore (x-4)(x+2) = 0 \quad \therefore x = 4 \quad \text{or} \quad x = -2$
Roots are real and distinct so type of closed form is: $u_n = A\alpha^n + B\beta^n$

Simultaneous equations: $\begin{array}{l} n = 0: \quad 2 = A(4)^0 + B(-2)^0 \\ n = 1: \quad 7 = A(4)^1 + B(-2)^1 \end{array} \quad \therefore \quad \begin{array}{l} A + B = 2 \\ 4A - 2B = 7 \end{array} \quad \therefore \quad A = \frac{11}{6} \quad \text{and } B = \frac{1}{6}$

Closed form: $u_n = \frac{1}{6}\left[11(4)^n + (-2)^n\right] \quad (n = 0, 1, 2, ...);$ 8th term: $u_8 = \frac{1}{6}\left[11(4)^8 + (-2)^8\right] =$ **120192**

Example 2
Find the closed form for the recurrence system: $u_0 = 2, u_1 = 7, u_{n+2} = 6u_{n+1} - 9u_n \quad (n = 0, 1, 2, ...)$ and use it to find the value of the 6th term.
Auxiliary equation: $x^2 - 6x + 9 = 0 \quad \therefore (x-3)(x-3) = 0 \quad \therefore x = 3 \quad \text{(repeated)}$
Roots are real and repeated so type of closed form is: $u_n = (A + Bn)\alpha^n$

Simultaneous equations: $\begin{array}{l} n = 0: \quad 2 = (A + B \times 0)(3)^0 \\ n = 1: \quad 7 = (A + B \times 1)(3)^1 \end{array} \quad \therefore \quad \begin{array}{l} A = 2 \\ 3A + 3B = 7 \end{array} \quad \therefore \quad A = 2 \quad \text{and } B = \frac{1}{3}$

Closed form: $u_n = (2 + \frac{1}{3}n)3^n \quad (n = 0, 1, 2, ...)$ so the 6th term is: $u_6 = (2 + \frac{1}{3} \times 6) \times 3^6 =$ **2916**

Exercises
For each of the following recurrence systems, find the closed form and the term indicated:

1 $u_0 = 5, \quad u_1 = 10, \quad u_{n+2} = u_{n+1} + 6u_n \quad (n = 0, 1, 2, ...)$ 8th term
2 $u_0 = 4, \quad u_1 = 7, \quad u_{n+2} = 5u_{n+1} - 6u_n \quad (n = 0, 1, 2, ...)$ 9th term
3 $u_0 = 3, \quad u_1 = -14, \quad u_{n+2} = -4u_{n+1} - 4u_n \quad (n = 0, 1, 2, ...)$ 5th term
4 $u_0 = 3, \quad u_1 = 7, \quad u_{n+2} = 5u_{n+1} + 6u_n \quad (n = 0, 1, 2, ...)$ 7th term
5 $u_0 = 5, \quad u_1 = 20, \quad u_{n+2} = -8u_{n+1} - 16u_n \quad (n = 0, 1, 2, ...)$ 4th term
6 $u_0 = 4, \quad u_1 = 6, \quad u_{n+2} = 1.8u_{n+1} - 0.77u_n \quad (n = 0, 1, 2, ...)$ 6th term

Answers
1 $4(3)^n + (-2)^n$, 8620 **2** $5(2)^n - 3^n$, -5281 **3** $(3+4n)(-2)^n$, 304
4 $\frac{1}{7}\left[10(6)^n + 11(-1)^n\right]$, 66653 **5** $(5-10n)(-4)^n$, 1600 **6** $8(1.1)^n - 4(0.7)^n$, 12.2118