The calculus power rules

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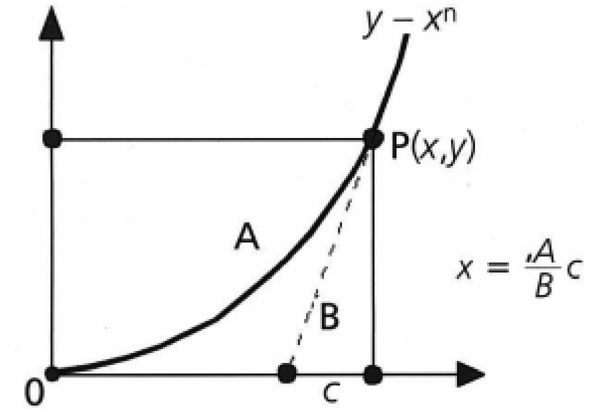
In *figure 1*, A and B are the areas of the regions of the rectangle split by the curve $y = x^n$.



An interactive version of this diagram at www.powermaths.org.uk makes it possible to verify the equation shown.[1] This curious result, which is true for all values of x and n, not merely those used in the investigation, is the basis for a useful pre-calculus exercise. The student can be prompted to deduce each calculus power rule from it, as in the routine shown below.

How can the area of the rectangle be expressed?

There are two ways: rectangle area $= A + B$, and also rectangle area $= xy = x \times x^n = x^{n+1}$.

Putting these together means that $A + B = x^{n+1}$.

Did you notice something about the ratio of the areas? It is always equal to the power of x.

So we have $\frac{A}{B} = n$; therefore, $A = Bn$, and we can substitute this into $A + B = x^{n+1}$.

Thus, $Bn + B = x^{n+1}$.

$\therefore B(n+1) = x^{n+1}$

$$\therefore B = \frac{x^{n+1}}{n+1}$$

You've probably seen this written formally as $\int x^n dx = \frac{x^{n+1}}{n+1}$, the integral power rule.

What do you know about the *gradient* of the curve at the point $P(x,y)$?

It is the same as the gradient of the tangent line at that point. But this is the hypotenuse of a right-angled triangle, so its gradient is $\frac{x^n}{c}$. Since $\frac{A}{B} = n$, by substitution into the equation $x = \frac{A}{B} \times c$ we get $x = n \times c$.

What then is the base of the triangle? By simple transposition, the base is $c = \frac{x}{n}$.

So the gradient of the tangent is

$$\frac{height}{base} = \frac{x^n}{\frac{x}{n}} = \frac{n}{x} \times x^n = nx^{n-1}.$$

You've probably seen this written formally as $\frac{dy}{dx} = nx^{n-1}$, the differential power rule.

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Note

1 Alternatively, there is a downloadable worksheet which verifies this for $0 < x < 10$ at www.powermaths.org.uk.