87.39 Tinkering with the calculus power rules

The graph of the power function $y = kx^n$, $x > 0$, $n > 0$ yields an intriguing piece of geometry in relation to the calculus power rules. Given the simplest form of the differential and integral power rules (omitting any coefficients and constants), it can be shown that the power 'n' is equal to A/B, where A and B are significant dimensions (Figure 1) or areas (Figure 2).

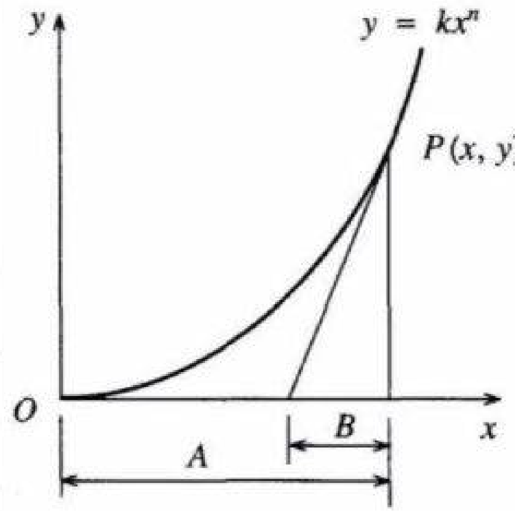

FIGURE 1

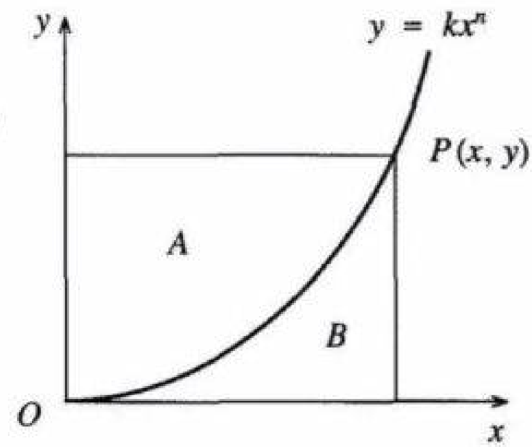

FIGURE 2

Figure 1: Let dimension A be the horizontal displacement x of point $P(x, y)$ on the curve of $y = kx^n$ and dimension B be the base length (on the x-axis) of a right triangle with apex at $P(x, y)$ and hypotenuse tangent to the curve. We note that, at $P(x, y)$, the gradient of the curve is the same as the slope of the hypotenuse which, by simple geometry, is equal to kx^n / B. Given that the differential power rule $\frac{dy}{dx} = knx^{n-1}$ describes the gradient of the curve at any point on the curve, then at $P(x, y)$ we have:

$$knx^{n-1} = \frac{kx^n}{B} \qquad \therefore B = \frac{kx^n}{knx^{n-1}} = \frac{x}{n} \qquad \therefore \frac{x}{B} = n \qquad \therefore \frac{A}{B} = n.$$

Figure 2: Let area A be the area of the region above, and area B the area of the region below, the curve $y = kx^n$, these regions being contained within the rectangle formed by lines drawn perpendicular to the co-ordinate axes from point $P(x, y)$ on the curve. The area of this rectangle is the product of the co-ordinates at $P(x, y)$ and since $y = kx^n$, this area is equal to kx^{n+1}. Given that, by the integral power rule $B = \int kx^n\,dx = \frac{kx^{n+1}}{n+1}$, we note that, since $\frac{kx^{n+1}}{n+1} + n\frac{kx^{n+1}}{n+1} = kx^{n+1}$, then $A = n\frac{kx^{n+1}}{n+1}$. Therefore $\frac{A}{B} = n$.

Note

When introducing calculus to sixth-formers, I sometimes got them to use calculator and graph paper to find the relationship between A, B and n

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